Nonnegative Edge Quasi-Consensus of Networked Dynamical Systems

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***Abstract*—Differing from the existing literature on node consensus, this brief studies consensus taking place on the edges of networked dynamical systems. A distributed edge quasi-consensus protocol is developed to lead the states of all edges to converge into a bounded region. For a connected network with nonnegative initial edge states, it is proved that the edge quasi-consensus can be reached while the states of all edges can be kept nonnegative. Moreover, the nonnegative edge quasi-consensus of networked systems with edge state constraints or input saturation is analyzed, respectively. Numerical simulations are provided to verify the effectiveness of the protocols.**

***Index Terms*—Edge dynamics, line graph, networked systems, nonnegative quasi-consensus.**

I. Introduction

CONSENSUS is a typical universal collective behavior in many networked engineering, social, and natural systems [1]–[11], which aims to guide all agents of a networked system to reach a common state. Owing to the graph theory [12], the consensus of multiagent systems or complex networks can be studied by characterizing the system topology as a graph, where nodes represent the agents and edges are the relationships among the agents. With the help of the graph theory, matrix theory, Lyapunov stability theory, and so on, the consensus of first-order systems [1], [2] and second-order ones [3]–[5], consensus on fixed [4]–[8], [11] and switching [2], [9], [10] topologies, and consensus without [2], [4], [10], [11] or with [5]–[9] leaders were extensively studied.

As is known, the existing works all focus on the node consensus. However, in reality, there are some physically meaningful quantities that are better described by edge dynamics, such as the data traffic in the Internet, flows of a certain material in a production–distribution system, the power flows in power networks, and the vehicle flows in transportation networks [13]. In 2012, Nepusz and Vicsek studied the controllability of edges on complex networks in [14] by endowing dynamics to edges. Also, in 2012, Slotine and Liu highlighted the significance of edge dynamics and further distinguished the difference between node controllability and edge controllability in [15]. Moreover, Lim and Ahn investigated the node consensus of systems with interconnection state saturation by entrusting dynamics to each edge in [13]. The importance of edge dynamics reminds us of the edge consensus, which aims at guiding the states of all edges to a common value. This kind of consensus behavior is very desirable in some real networks. For example, it was shown in [16] that the pattern of communication is the most important predictor of a team's success. In particular, the frequency of communications between each pair of members in a high-performance team reaches a consensus. In detail, the frequency of communications between any two members is precisely the weight of an edge. Therefore, to build an efficient team, the weight of communication between any two members should tend to a same value with the others so as to have a balanced high efficiency. Another example is about the power grid. In the power grid, the current power flow on each line can be described by edge dynamics, which is bounded by its maximum capacity. Moreover, in practical engineering, to enhance the stable operation of the transmission grid, it is desirable that the ratio between the current power flow of each line with its maximum capacity turns to the same optimal value. Similarly, in transportation networks, for the purpose of improving the road utilization rate and avoiding traffic jams, the ratio between the current vehicle flow of each road with its maximum capacity would be better to reach the same. On the other hand, note that the number of information or work exchanges in social networks, the current power flows of lines in the power grids, and the current vehicle flows of roads in the transportation networks are all physically meaningful variables and they generally must be nonnegative to make sense of the world. Thus, during the process toward edge consensus, the values of all edges are required to be nonnegative in this brief.

Motivated by the aforementioned discussions, this brief investigates the nonnegative edge quasi-consensus of undirected networks, which means that the states of all edges should enter into and then remain therein a bounded region while keeping the states nonnegative if the initial states are nonnegative.

The significance and contribution of this brief are threefold. First, a distributed edge protocol is proposed for the first time to direct all edges to achieve nonnegative quasi-consensus. Second, nonnegative edge quasi-consensus for networked systems with

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a state constraint is solved. Rigorous theoretical analysis shows that, for nonnegative initial states within the constraint, the states of all edges can converge to a bounded region while keeping the states of all edges nonnegative and within the constraint. Finally, the nonnegative edge quasi-consensus of connected networks with input saturation is proved reachable for initial edge states selected from any bounded region by choosing the network coupling strength properly.

Notation: Let Z be the set of nonnegative integers. For any k, $n \in Z$ with $k \leq n$, $[k, n]$ is the set of integers $\{k, k+1, k+2, \ldots, n\}$. R is the set of real numbers. A real matrix (or a real vector) A with all entries nonnegative is denoted by $A \geq 0$. Moreover, the (i, j)th entry of matrix A is denoted by A_{ij}, while $A_{[i,:]}$ is the ith row of A. $\mathbf{0}_{M\times M}$ is a matrix with all entries being zero. $\|\cdot\|$ is the 2-norm. For any scalar $x \in R$, $|x|$ is its absolute value.

II. Model Description

A. Model Description

As discussed earlier, for a network, an important issue is how to make the interactions between any two nodes be relatively equal—in other words, how to force the network to achieve edge consensus. This brief only studies the case where the edges represent some physically meaningful quantities with nonnegative characteristics such as the data traffic in the Internet, which requires keeping the states of the edges nonnegative. In particular, as in the Internet, the state of an edge equaling zero means that there is no data traffic on this edge.

Consider an undirected network described by a triple $\mathcal{G} = (V, E, A)$ with node set $V = \{1, 2, \ldots, N\}$ and edge set $E = \{(i, j)|$ if there exists an edge between node i and node $j\}$ as well as an adjacency matrix $A = (a_{ij}) \in R^{N\times N}$ with $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise. The Laplacian matrix of $\mathcal{G}$ is $L = D - A = (l_{ij}) \in R^{N\times N}$, where $D = \mathrm{diag}\{d_1, d_2, \ldots, d_N\}$ is the degree matrix of $\mathcal{G}$ with $d_i = \sum_{j=1, j\neq i}^{N} a_{ij}$.

The investigation is begun with associating each edge with a state variable. Let $x_{ij}(t) \in R$ be the state of the edge between node i and node j at time t, $i < j$. Obviously, there are $M = (1/2)\sum_{i=1}^{N} d_i$ undirected edges in $\mathcal{G}$, and each edge regulates itself according to the following dynamical equation:

$$\dot{x}_{ij}(t) = \sum_{k=1, k\neq j}^{N} a_{ik} f_{ik}(\alpha)\left[x_{ik}(t) - x_{ij}(t)\right] + \sum_{s=1, s\neq i}^{N} a_{js} f_{js}(\alpha)\left[x_{js}(t) - x_{ij}(t)\right], i, j, k, s \in [1, N] \quad (1)$$

with $\alpha > 0$ being a given threshold and

$$f_{ik}(\alpha) = \begin{cases} 1, & \text{if } |x_{ik}(t) - x_{ij}(t)| > \alpha \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

As shown in Fig. 1, $x_{ij}(t)$ depends only on the states of itself and its neighboring edges. Here, two edges are said to be neighboring if and only if they have a common node. Equation (1) describes the distributed communications between each edge and its neighboring edges. Equation (2) shows that the updating of $x_{ij}(t)$ will not be affected by its neighboring edge x_{ik} (or x_{js}) if the difference between them is not larger than α. In other words, two neighboring edges are regarded to have reached a consensus if the difference between them are not larger than the given positive constant α.

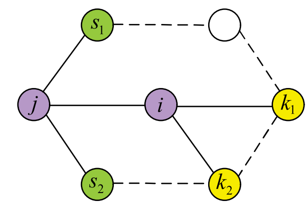

Fig. 1. Illustration on the distributed interaction relationship among edges.

Definition 1: System (1) is said to achieve nonnegative edge quasi-consensus with respect to a bound $\gamma > 0$ if

$$\lim_{t\to\infty} |x_{ij}(t) - x_{rs}(t)| \leq \gamma$$
$$x_{ij}(t) \geq 0 \text{ for } t \geq 0, \forall\, i, j, r, s \in [1, N]$$

for any nonnegative initial state conditions. In particular, if $\gamma = 0$, the nonnegative edge quasi-consensus becomes nonnegative edge consensus.

B. Model Reformulation Based on Line Graph

Define a state vector $Y \in R^{M\times 1}$ as

$$Y = (y_1 \quad y_2 \quad \cdots \quad y_M)^T \quad (3)$$

with each element in Y corresponding to an undirected edge in $\mathcal{G}$. Then, the adjacencies among elements in Y will be described with a line graph as follows.

First, for network $\mathcal{G}$, its line graph, denoted as $\mathcal{L}(\mathcal{G})$, describes the adjacencies among the edges of $\mathcal{G}$. For better understanding, hereinafter, call $\mathcal{G}$ the original network, while $\mathcal{L}(\mathcal{G})$ is the line graph. Moreover, the adjacency matrix and Laplacian matrix of the line graph $\mathcal{L}(\mathcal{G})$ are denoted as $\breve{G} = (\breve{g}_{ij}) \in R^{M\times M}$ and $\breve{L} = (\breve{l}_{ij}) \in R^{M\times M}$, respectively. Here, $\breve{G}$ describes the connections of the edges in its line graph $\mathcal{L}(\mathcal{G})$, which is determined by the topology of $\mathcal{G}$ and is called the physical adjacency matrix. Similarly, $\breve{L}$ is named the physical Laplacian matrix. However, the interactions of edges do not exactly follow the physical adjacency matrix $\breve{G}$ but the interaction adjacency matrix $\breve{G}(\alpha)$, where $\breve{G}(\alpha) = (\breve{g}_{ij}(\alpha)) \in R^{M\times M}$ with $\breve{g}_{ij}(\alpha) = 1$ if there is an edge between y_i and y_j in $\mathcal{L}(\mathcal{G})$ and simultaneously $|y_i - y_j| > \alpha$; otherwise, $\breve{g}_{ij}(\alpha) = 0$. Accordingly, its interaction Laplacian matrix is $\breve{L}(\alpha) = (\breve{l}_{ij}(\alpha)) \in R^{M\times M}$ with $\breve{l}_{ij}(\alpha) = -\breve{g}_{ij}(\alpha)$ if $i \neq j$ and $\breve{l}_{ij}(\alpha) = \sum_{j=1, j\neq i}^{N} \breve{g}_{ij}(\alpha)$ if $i = j$. By the definition of $\breve{L}(\alpha)$, protocol (1) with $f_{ik}(\alpha)$ in (2) can be written into a compact form

$$\dot{Y}(t) = -\breve{L}(\alpha)Y(t). \quad (4)$$

III. Main Results

To start with, some preliminaries are introduced.

Definition 2 [17]: A Metzler matrix is a real square matrix with nonnegative off-diagonal entries.

Definition 3 [17]: System $\dot{x}(t) = Ax(t)$ is called positive if, for all initial condition $x(0) \geq 0$, the state trajectory satisfies $x(t) \geq 0$ for all $t \geq 0$.

Remark 1: In conformity with the fact that the value of each variable in system (1) is nonnegative, the positive system in [17] is renamed as nonnegative system in this brief.

Lemma 1 [17]: System $\dot{x}(t) = Ax(t)$ is positive if and only if A is Metzler.

Lemma 2 [18]: The line graph of an undirected connected graph is also undirected and connected.

Lemma 3 [1]: Let G be a connected graph, and suppose that each node of G is described by the distributed linear protocol $\dot{x} = -Lx$ with L being the Laplacian matrix of G. Then, all the nodes of the graph globally asymptotically reach the average value of the initial values.

The definition of $\breve{L}(\alpha)$ shows that $-\breve{L}(\alpha)$ is a Metzler matrix. Based on Lemma 1 and the analysis in [19], for nonnegative initial states, all states of network (4) can be kept nonnegative (irrespective of the network being stable or not). The following theorem is one main result of this brief.

Theorem 1: Consider an undirected connected network with N nodes and M edges, where each edge is steered by protocol (1). Then, the nonnegative edge quasi-consensus with respect to the bound γ can be achieved if

$$(M-1)\alpha \leq \gamma. \tag{5}$$

That is, for any $x_{ij}(0) \geq 0$, one has $\lim_{t\to\infty} |x_{ij}(t) - x_{rs}(t)| \leq \gamma$, $\forall\, i, j, r, s \in [1, N]$ and $x_{ij}(t) \geq 0$ for all $t \geq 0$.

Proof: First, the Metzler property of $-\breve{L}(\alpha)$ guarantees that, if the initial states are nonnegative, then all states of the edges will remain nonnegative.

Next, it will be proved that system (1) can achieve nonnegative quasi-consensus.

Define $v_i = \sum_{j=1}^{M} \breve{g}_{ij}(\alpha)[y_i(t) - y_j(t)]$ so that

$$v = \breve{L}(\alpha)Y(t). \tag{6}$$

Construct a Lyapunov function candidate as

$$V = \frac{1}{2}v^T(t)v(t). \tag{7}$$

Taking the derivative of V along with (6) yields

$$\begin{aligned} \dot{V} &= v^T(t)\breve{L}(\alpha)\left(-\breve{L}(\alpha)Y(t)\right) \\ &= -v^T(t)\breve{L}(\alpha)v(t) \\ &\leq 0. \end{aligned} \tag{8}$$

Equation (8) implies that $\dot{V}(t) = 0$ if $\breve{L}(\alpha) = \mathbf{0}_{M\times M}$ or $v(t) = \mathbf{0}_{M\times 1}$. Therefore, $\|v(t)\|$ will decrease, and more elements in $\breve{L}(\alpha)$ will tend to 0. On the other hand, the definition of $\breve{L}(\alpha)$ implies that $\dot{V}(t) = 0$ if and only if $\breve{L}(\alpha) = \mathbf{0}_{M\times M}$. Since the original network $\mathcal{G}$ is connected, Lemma 2 implies that $\mathcal{L}(\mathcal{G})$ is always connected, so $\breve{L}(\alpha) = \mathbf{0}_{M\times M}$ if and only if $|x_{ik}(t) - x_{ij}(t)| \leq \alpha$ and $|x_{js}(t) - x_{ij}(t)| \leq \alpha$ for all $i, j, k, s \in [1, N]$. Hence, by LaSalle's invariance principle, it follows that $\lim_{t\to\infty} |x_{ik}(t) - x_{ij}(t)| \leq \alpha$ and $\lim_{t\to\infty} |x_{js}(t) - x_{ij}(t)| \leq \alpha$ for all $i, j, k, s \in [1, N]$, i.e., all edges will remain inside the bounded region with respect to the bound $(M-1)\alpha$. Then, by (5), the states of all edges will converge into a bounded region with respect to the bound γ. ■

In reality, it is impractical to assume that edges possess infinite transmission capacity, i.e., the state of each edge is generally constrained into a bounded interval. Assume that the constraint on each edge is Δ, i.e.,

$$y_i \leq \Delta, \quad i \in [1, M]. \tag{9}$$

In the following, it will be proved that protocol (1) can direct all edges to enter into and then remain inside a bounded region while keeping the state of each edge within $[0, \Delta]$ if all the initial states are located inside $[0, \Delta]$.

Theorem 2: Consider an undirected connected network $\mathcal{G}$ with N nodes and M edges. Each edge evolves according to protocol (1). Then, for $x_{ij}(0) \in [0, \Delta]$, the states of all edges can be kept in the bounded region $[0, \Delta]$ and can achieve nonnegative quasi-consensus. That is, for $x_{ij}(0) \in [0, \Delta]$ and α as in (5), all edges will enter into and then remain inside a bounded region with respect to the bound γ while keeping the states of all edges within $[0, \Delta]$.

Proof: First, the dynamics of the system $\dot{Y}(t) = -\breve{L}(\alpha)Y(t)$ in (4) and the definition of $\breve{L}(\alpha)$ show that the states of all edges can be kept nonnegative for $t \geq 0$ if the initial values are nonnegative.

By (4), one obtains

$$Y(t) = e^{-\breve{L}(\alpha)t}Y(0) \tag{10}$$

where $e^{-\breve{L}(\alpha)t}$ is a stochastic matrix for all $t \geq 0$, whether $\breve{L}(\alpha)$ is irreducible or not. Therefore, for $y_i(0) \in [0, \Delta]$, one has $y_i(t) = [e^{-\breve{L}(\alpha)t}]_{[i,:]}Y(0) \leq \Delta$, which implies that the states of all edges will be kept within $[0, \Delta]$ for $t \geq 0$.

In addition, by Theorem 1, all edges will converge to a bounded region with respect to the bound γ. ■

Corollary 1: If $\alpha = 0$, protocol (1) becomes

$$\begin{aligned} \dot{x}_{ij}(t) = &\sum_{k=1,k\neq j}^{N} a_{ik}\left[x_{ik}(t) - x_{ij}(t)\right] \\ &+ \sum_{s=1,s\neq i}^{N} a_{js}\left[x_{js}(t) - x_{ij}(t)\right],\ i, j, k, s \in [1, N]. \end{aligned} \tag{11}$$

Then, the connected network can achieve nonnegative edge average consensus under protocol (11). That is, the network can achieve nonnegative edge consensus to the average value of the initial states. Moreover, if the initial states of all edges are located within a bounded interval $[0, \Delta]$, then the connected network can reach nonnegative edge consensus while keeping the states of all edges to be with $[0, \Delta]$ for $t \geq 0$.

Proof: If $\alpha = 0$, the interacting topology is the physical topology and $\breve{L}(\alpha) = \breve{L}$ is always connected. By Lemma 3 together with Theorem 1 and Theorem 2, one can directly verify Corollary 1. ■

IV. Nonnegative Edge Consensus of Networks With Input Saturation

Input saturation is a typical nonlinearity induced by the physical device. Theorem 2 motivates us to address the nonnegative edge quasi-consensus of a network with input saturation, which means that the magnitude of the control input $u_{ij}(t)$ is constrained in a bounded interval $[-\omega, \omega]$. In this case, the dynamics of each edge are described by

$$\begin{aligned} \dot{x}_{ij}(t) &= \mathtt{sat}\left(u_{ij}(t)\right), \quad i, j, k, s \in [1, N] \\ u_{ij}(t) &= \sum_{k=1,k\neq j}^{N} a_{ik}f_{ik}(\alpha)\left[x_{ik}(t) - x_{ij}(t)\right] \\ &\quad + \sum_{s=1,s\neq i}^{N} a_{js}f_{js}(\alpha)\left[x_{js}(t) - x_{ij}(t)\right] \end{aligned} \tag{12}$$

with

$$\mathtt{sat}(u_{ij}) = \begin{cases} \omega, & \text{if } u_{ij} \geq \omega \\ u_{ij}(t), & \text{if } -\omega < u_{ij} < \omega \\ -\omega, & \text{if } u_{ij} \leq -\omega. \end{cases} \quad (13)$$

Here, ω is the saturation level. To avoid any meaningless discussion, presume $\omega > \alpha$.

The following theorem is another main result of this brief.

Theorem 3: For an undirected connected network of N nodes and M edges, in which each edge regulates itself according to (12), if all the initial values of edges are located inside the bounded interval $[0, \omega/(M-1)]$, then the network can reach nonnegative edge quasi-consensus with respect to the bound γ for $(M-1)\alpha \leq \gamma$. That is, for $x_{ij}(0) \in [0, \omega/(M-1)]$, one has $\lim_{t\to\infty} |x_{ij}(t) - x_{rs}(t)| \leq \gamma, \forall\, i, j, r, s \in [1, N]$ and $x_{ij}(t) \geq 0$ for $t \geq 0$.

Proof: Denote $\mathcal{U}(t) = [U_1(t), U_2(t), \ldots, U_M(t)]^T = -\check{L}(\alpha)Y(t)$. The preconditions $x_{ij}(0) \in [0, \omega/(M-1)]$ lead to

$$|U_i(0)| = \left| -\check{L}(\alpha)_{[i,:]} Y(0) \right| < \omega$$

and the following discussion focuses on the proof of $|U_i(t)| < \omega$ for $t > 0$ and $i \in [1, M]$.

By the definition of $u_{ij}(t)$ in (12), one has

$$\mathcal{U}(t) = -\check{L}(\alpha)Y(t) = -\check{L}(\alpha)\left[e^{-\check{L}(\alpha)t} Y(0)\right].$$

Then, the facts that $e^{-\check{L}(\alpha)t}$ is a stochastic matrix for $t \geq 0$ and that $x_{ij}(0) \in [0, \omega/(M-1)]$ together imply that

$$|U_i(t)| = \left| -\left[\check{L}(\alpha)\right]_{[i,:]} \left[e^{-\check{L}(\alpha)t} Y(0)\right] \right| < \omega, i \in [1, M].$$

That is, $x_{ij}(0) \in [0, \omega/(M-1)]$ can guarantee that $|u_{ij}(t)| < \omega$ for $t \geq 0$, and the input saturation can be avoided. Consequently, by Theorem 1 and Theorem 2, protocol (12) can direct all edges to achieve nonnegative quasi-consensus with respect to the bound γ if $(M-1)\alpha < \gamma$. ■

Corollary 2: If $\alpha = 0$, the protocol with input saturation (12) becomes

$$\begin{aligned} \dot{x}_{ij}(t) &= \mathtt{sat}\left(u_{ij}(t)\right), \quad i, j, k, s \in [1, N] \\ u_{ij}(t) &= \sum_{k=1, k\neq j}^{N} a_{ik}\left[x_{ik}(t) - x_{ij}(t)\right] \\ &\quad + \sum_{s=1, s\neq i}^{N} a_{js}\left[x_{js}(t) - x_{ij}(t)\right] \end{aligned}$$

with $\mathtt{sat}(\cdot)$ defined in (13), which can ensure all edges reaching nonnegative edge consensus by properly selecting the initial states.

Proof: By Corollary 1 and Theorem 2 as well as Theorem 3, it is straightforward to obtain Corollary 2. ■

Remark 2: To investigate networks with input saturation, it is of great importance to enlarge the set of the initial states. If each edge updates itself by following

$$\begin{aligned} \dot{x}_{ij}(t) &= \mathtt{sat}\left(\bar{u}_{ij}(t)\right), \quad i, j, k, s \in [1, N] \\ \bar{u}_{ij}(t) &= \varepsilon u_{ij}(t) \end{aligned} \quad (14)$$

with $\varepsilon > 0$ being the coupling strength and $u_{ij}(t)$ as in (12), then, for any bounded interval $[0, \Delta]$, the network can reach nonnegative edge quasi-consensus with respect to the bound γ if $\varepsilon < (\omega/(M-1)\Delta)$. Here, the ε plays the same role of the low-gain matrix of $P(\varepsilon)$ in [6] and [7].

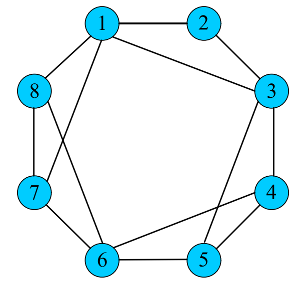

Fig. 2. Topology of the original network in simulation.

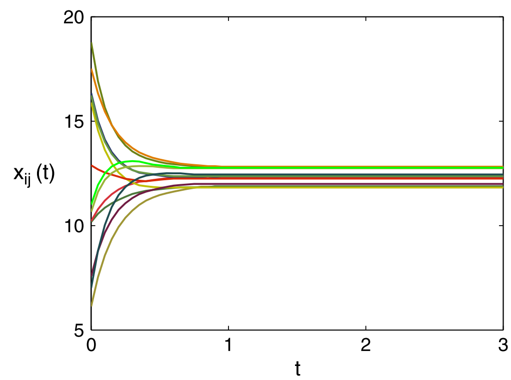

Fig. 3. Nonnegative edge quasi-consensus of the network without edge state saturation under protocol (1).

Remark 3: Equation (5) implies that one can direct all edges to enter into and then remain in a bounded region with respect to the given bound γ by properly selecting the parameter α. Moreover, for the chosen α satisfying $(M-1)\alpha \leq \gamma$, the value γ is actually an upper bound of the edge convergence region.

V. Numerical Simulations

In this section, some numerical simulations are presented to verify the theoretical results of this brief.

The topology of the original network is shown in Fig. 2, which contains $N = 8$ nodes and $M = 13$ edges.

For nonnegative initial states randomly selected from [0, 20], with a given threshold $\alpha = 0.5$, Fig. 3 shows that, under protocol (1), the states of all edges remain to be nonnegative for $t > 0$ and all edges finally enter into and remain inside a bounded region. Here, the bound of the bounded region is $\max - \min = 11.0026 - 9.7759 = 1.2268$, while $(M-1)\alpha = 6$, and obviously, $(M-1)\alpha < \gamma$ is a sufficient condition for $\max - \min < \gamma$.

Fig. 4 shows the edge state convergence of the same network with edge state saturation, where each edge is governed by protocol (1). Here, $\Delta = 3$, and $\alpha = 0.1$. Fig. 4 shows that, with the edge state saturation (9), the states of all edges remain in the bounded region while keeping all the states in [0, 3] if the initial states of all edges are selected from [0, 3]. Also, the bound of the bounded region is $\max - \min = 1.6348 - 1.4064 = 0.2284$, and $(M-1)\alpha = 1.2000$.

In the simulations on nonnegative edge quasi-consensus of the network subject to input saturation, the saturation level is $\omega = 2$. The initial states of all edges are chosen from $[0, (\omega/(M-1))] = [0, 1/6]$ according to *Theorem* 3. Here, $\alpha = 0.01$.

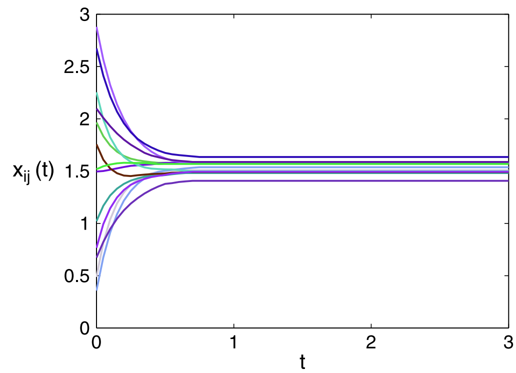

Fig. 4. Nonnegative edge quasi-consensus of the network with edge state saturation (9) under protocol (1).

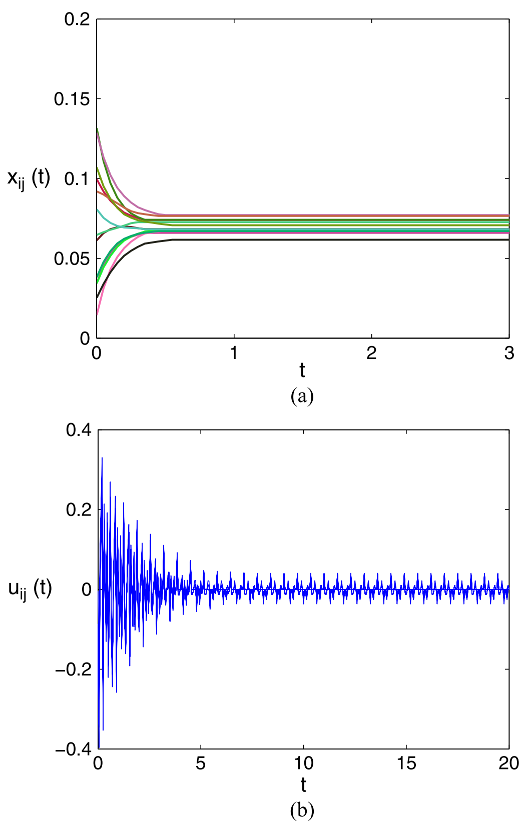

Fig. 5. Nonnegative edge quasi-consensus of the network with input saturation under protocol (12). (a) Edge state convergence. (b) Control input.

Fig. 5(a) shows that all edges can remain in the bounded region with bound $\max - \min = 0.0856 - 0.0647 = 0.0209$, and $(M-1)\alpha = 0.1200$. In Fig. 5(b), the control inputs u_{ij} meeting $|u_{ij}| < \omega$ are all bounded in $[-0.5, 0.5]$.

VI. Conclusion

The nonnegative edge quasi-consensus of networked systems without constraints on undirected networked systems and that with constraints on undirected networked systems were both investigated in detail. A distributed algorithm was proposed, in which each edge will not update itself until the differences between itself and its neighboring edges are larger than a given threshold. It can guarantee that the states of all edges will be kept nonnegative if all the initial states are nonnegative, and all edges can remain in a bounded region. Theoretical analysis was given to the nonnegative edge quasi-consensus of undirected networks with a state constraint and input saturation, respectively. It was shown that the network subject to state and control input constraints can ensure all edges reaching nonnegative quasi-consensus. Future efforts will be focused on the nonnegative edge consensus of more general dynamical systems with different complex topologies.

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