Stability and Bifurcation of Delayed Fractional-Order Dual Congestion Control Algorithms

Min Xiao, *Member, IEEE*, Wei Xing Zheng, *Fellow, IEEE*, Guoping Jiang, *Senior Member, IEEE*, and Jinde Cao, *Fellow, IEEE*

***Abstract*—In this technical note, fractional-order congestion control systems are introduced for the first time. In comparison with the conventional integer-order dual congestion control algorithms, the fractional control algorithms are more accurate and versatile. Bifurcation theory in fractional-order differential equations is still an outstanding problem. Sufficient conditions for the occurrence of Hopf bifurcations are extended from integer-order dynamical systems to fractional-order cases. Then, these conditions are used to establish the existence of Hopf bifurcations for the delayed fractional-order model of dual congestion control algorithms proposed in this note. Finally, the onsets of bifurcations are identified, where Hopf bifurcations occur and a family of oscillations bifurcate from the equilibrium. Illustrative examples are also provided to demonstrate the theoretical results.**

***Index Terms*—Congestion control, fractional-order dynamical systems, hopf bifurcation, stability.**

I. Introduction

Research in dynamics and control of congestion in the Internet has achieved considerable progresses based on integer-order differential equations during the past two decades [1]–[5]. However, its applications to real practical congestion systems are still limited due to the lack of more accurate models of congestion control algorithms.

Fractional calculus is commonly believed to have stemmed from a question raised in the year 1695 by de L'Hospital in a letter to Leibniz, but its application to physics and engineering has been reported only in recent years. It has been found that in many practical cases, systems can be more adequately described by fractional-order differential equations. For instance, fractional-order models have been proposed to model a wide range of problems in electrical capacitors [6], viscoelastic materials [7], finance systems [8], transmission lines [9], and neurons [10]. Also, in recent years, considerable attention has been paid to using the great potential of fractional calculus in system control issues, such as system approximation, system sampling, and design and implementation of controllers [11]–[15].

Fractional-order derivatives provide an excellent instrument for the description of memory and hereditary properties of various processes due to the existence of a memory term in the model [16]. It has been shown that for congestion control algorithms, the price at the link representing the congestion level is the solution of a delayed fractional-order differential equation [17]. Unlike integer-order differential equations, fractional-order differential equations are naturally related to memory effects and long-range dispersion processes, which exist in most congestion control systems. Therefore, the fractional-order model is more accurate than the classical integer-order model when modeling congestion control algorithms.

One of the important properties of congestion control algorithms is the stability. Sufficient conditions of stability have been obtained for integer-order congestion systems [1]–[4]. However, it was found in [18] that some commonly active queue management schemes coupled with the current congestion avoidance transmission control protocol (TCP) algorithm may lose the local stability due to an increase in delays or capacity, or a decrease in the number of connections. The loss of stability causes some nonlinear dynamical behaviors, such as chaos and bifurcation. Thus, in addition to an investigation of stability, Hopf bifurcation and control have also begun to draw much attention for delayed integer-order congestion systems [19]–[22].

However, owing to limitations of the existing theories, few studies of Hopf bifurcations for delayed fractional-order congestion systems have been reported. It is worth mentioning that the qualitative theory of Hopf bifurcations for the case of fractional-order dynamical systems has not completely settled yet. Based on the observations arising from numerical simulations, the conditions for the occurrence of Hopf bifurcations were only proposed for fractional-order dynamical systems without time delays [23], [24]. It should be noted that a fractional-order congestion control model was introduced in [17]. The delayed fractional-order congestion control model can exhibit a Hopf bifurcation (i.e., periodic oscillations appear) as the delay passes through the critical value. Motivated by the above discussions, this note will establish some bifurcation conditions for delayed fractional-order dynamical systems by choosing the system parameter as the bifurcation parameter and will be devoted to investigate the stability and bifurcations for a delayed fractional-order congestion control system.

This note is organized as follows. Section II summarizes some preliminaries of fractional-order systems, establishes sufficient conditions for the occurrence of Hopf bifurcations for general one-dimensional (1-D) delayed fractional-order systems, and proposes a fractional-order system of dual congestion control algorithms. Section III presents the main theorems of stability and Hopf bifurcations for the delayed

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M. Xiao and G. Jiang are with the College of Automation, Nanjing University of Posts and Telecommunications, Nanjing 210003, China (e-mail: candymanxm2003@aliyun.com; jianggp@njupt.edu.cn).

W. X. Zheng is with the School of Computing, Engineering and Mathematics, Western Sydney University, Sydney, NSW 2751, Australia (e-mail: w.zheng@westernsydney.edu.au).

J. Cao is with the Research Center for Complex Systems and Network Sciences, Southeast University, Nanjing 210096, China (e-mail: jdcao@seu.edu.cn).

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fractional-order congestion control system. Section IV concludes the presentation.

II. Preliminaries

A. Delayed Fractional-Order Systems

Definition 1 ([16]): For a continuous function f, with $f' \in L^1(\mathbb{R}^+)$, the Caputo fractional derivative operator of order $\alpha \in (0,1)$ of f is defined in the following form:

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function.

Remark 1: When $\alpha \to 1^-$, the fractional-order derivative ${}_0^C D_t^\alpha f(t)$ converges to the integer-order derivative $f'(t)$ [25].

Remark 2: Equation (1) defines the classical uninitialized fractional derivative. The definition of initialized fractional-order derivatives can be found in [26] and [27].

Consider the n-dimensional linear fractional-order system with multiple time delays:

$$ {}_0^C D_t^\alpha x_i(t) = \sum_{j=1}^n a_{ij} x_j(t-\tau_{ij}), \quad i = 1,\ldots,n \tag{2}$$

where $\alpha \in (0,1)$ for $i = 1,\ldots,n$. The initial conditions are $x_i(t) = \phi_i(t), t \in [-\tau_{\max}, 0]$ for some continuous function $\phi_i(t)$, where $\tau_{\max} = \max_{1\le i,j\le n}\{\tau_{ij}\}$. The stability of the zero solution of system (2) depends on the distribution of the roots of the associated characteristic equation (3) as given below:

$$ \det \begin{pmatrix} s^\alpha - a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \cdots & -a_{1n}e^{-s\tau_{1n}} \\ -a_{21}e^{-s\tau_{21}} & s^\alpha - a_{22}e^{-s\tau_{22}} & \cdots & -a_{2n}e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}e^{-s\tau_{n1}} & -a_{n2}e^{-s\tau_{n2}} & \cdots & s^\alpha - a_{nn}e^{-s\tau_{nn}} \end{pmatrix} = 0. \tag{3}$$

Remark 3: Equation (2) describes the traditional fractional-order system without considering the pseudostate value. A pseudostate space description of fractional-order systems can be found in [28]–[30].

Theorem 1 ([31]): If all the roots of the characteristic equation (3) have negative real parts, then the zero solution of system (2) is Lyapunov globally asymptotically stable.

Remark 4: Theorem 1 indicates that the stability boundary for the delayed fractional-order system (2) is the imaginary axis.

Remark 5: If $\tau_{ij} = 0, i,j = 1,\ldots,n$, then Theorem 1 converts into Matignon criterion [34]: if all the roots λs of the equation $\det(\lambda I - A) = 0$ satisfy $|\arg(\lambda)| > \alpha\pi/2$, then the zero solution of system (2) is Lyapunov globally asymptotically stable, where $A = (a_{ij})_{n\times n}$ is the coefficient matrix and $\lambda = s^\alpha$. It can be seen that the stability boundary is described by $|\arg(\lambda)| = \alpha\pi/2$ (or $|\arg(s)| = \pi/2$) for the fractional-order system (2) without delays.

Remark 6: If all the eigenvalues λs of A satisfy $|\arg(\lambda)| > \alpha\pi/2$, and the characteristic equation (3) has no purely imaginary roots for any $\tau_{ij} > 0, i,j = 1,\ldots,n$, then the zero solution of system (2) is Lyapunov globally asymptotically stable (see [31, Corollary 3])

B. Stability and Hopf Bifurcation of 1-D Delayed Fractional-Order Systems

There are some fundamental differences between the dynamical behaviors of fractional-order and integer-order systems. One of the fundamental differences is that unlike integer-order systems, the oscillatory responses of Caputo-based fractional-order systems cannot be exactly periodic, but they can approach periodic solutions as time tends to infinity [32], [33].

It is well known that the Hopf bifurcation is the birth of a limit cycle from an equilibrium in integer-order dynamical systems, when the equilibrium changes stability via a pair of purely imaginary eigenvalues. However, the qualitative theory of Hopf bifurcations for fractional-order dynamical systems is still an open question. Based on the observations from numerical simulations, the conditions for the occurrence of Hopf bifurcations were proposed in [23] and [24] for fractional-order dynamical systems without time delays, but were not proved there. Furthermore, the conditions for the occurrence of Hopf bifurcations for delayed fractional-order systems have not been reported yet. In this section, we put forward the conditions for the occurrence of Hopf bifurcations for 1-D delayed fractional-order systems.

Consider the following 1-D delayed fractional-order system:

$$ {}_0^C D_t^\alpha x(t) = f(x(t-\tau);\mu) \tag{4}$$

where $\alpha \in (0,1)$ and μ is the bifurcation parameter. Suppose that system (4) has an equilibrium point 0.

Theorem 2: If $f'(0;\mu) < 0$ and $[-f'(0;\mu)]^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2], j \in \mathbb{Z}$, then the zero solution of (4) is Lyapunov locally asymptotically stable.

Proof: The linearized system of (4) is given by

$$ {}_0^C D_t^\alpha x(t) = f'(0;\mu) x(t-\tau) \tag{5}$$

with the characteristic equation

$$ s^\alpha - f'(0;\mu) e^{-s\tau} = 0. \tag{6}$$

Let $s = \mathrm{i}\omega = \omega(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2})(\omega > 0)$ be a root of (6). Then,

$$ \omega^\alpha \left(\cos\frac{\alpha\pi}{2} + \mathrm{i}\sin\frac{\alpha\pi}{2}\right) - f'(0;\mu)(\cos\omega\tau - \mathrm{i}\sin\omega\tau) = 0.$$

Separating the real and imaginary parts gives

$$ \omega^\alpha \cos(\alpha\pi/2) - f'(0;\mu)\cos\omega\tau = 0$$
$$ \omega^\alpha \sin(\alpha\pi/2) + f'(0;\mu)\sin\omega\tau = 0.$$

Hence,

$$ (\omega^\alpha)^2 + f'^2(0;\mu) - 2f'(0;\mu)\omega^\alpha \cos((\alpha\pi/2) + \omega\tau) = 0. \tag{7}$$

Noting that $f'(0;\mu) < 0$, it follows that

$$\begin{aligned} & (\omega^\alpha)^2 + f'^2(0;\mu) - 2f'(0;\mu)\omega^\alpha \cos((\alpha\pi/2) + \omega\tau) \\ \ge\ & (\omega^\alpha)^2 + f'^2(0;\mu) + 2f'(0;\mu)\omega^\alpha \\ =\ & (\omega^\alpha + f'(0;\mu))^2 . \end{aligned}$$

Obviously, if $[-f'(0;\mu)]^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$, then (7) has no positive real roots, meaning that (6) has no purely imaginary roots with positive imaginary parts.

Let $s = -\mathrm{i}\omega = \omega[\cos\frac{\pi}{2} + \mathrm{i}\sin(-\frac{\pi}{2})](\omega > 0)$ be a root of (6). Similarly, it can be proved that (6) has no purely imaginary roots with negative imaginary parts under the assumption $[-f'(0;\mu)]^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$.

To sum up the above arguments, if $[-f'(0;\mu)]^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$, then the characteristic equation (6) has no purely imaginary roots.

On the other side, it can be seen that the coefficient matrix of the linearized system (5) has one eigenvalue $\lambda = f'(0;\mu) < 0$ satisfying $|\arg(\lambda)| > \alpha\pi/2$. Applying Remark 6, the zero solution of (4) is Lyapunov locally asymptotically stable. ∎

Theorem 3: If the following conditions hold

1) $\mu = \mu_j$, where μ_j is the root of the equation

$$[-f'(0;\mu)]^{1/\alpha} = \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2], \quad j \in \mathbb{Z};$$

2) $\frac{d\mathrm{Re}[s(\mu)]}{d\mu}\Big|_{\mu=\mu_j} \neq 0$, where $s(\mu)$ is the root of the characteristic equation (6) and $\mathrm{Re}\{\cdot\}$ denotes the real parts of the complex eigenvalues;

then the delayed fractional-order system (4) undergoes a Hopf bifurcation at the zero solution.

Proof: Theorem 1 indicates that the stability margin for delayed fractional-order systems is the imaginary axis. Therefore, when a complex eigenvalue of the characteristic equation crosses the stability boundary: the imaginary axis, a Hopf bifurcation occurs in system (4), where a branch of periodic oscillations appears from the zero solution.

From the proof of Theorem 2, Condition 1) implies that when $\mu = \mu_j$, the characteristic equation (6) has a pair of purely imaginary roots $\pm \mathrm{i}\omega_0$, where $\omega_0 = [-f'(0;\mu_j)]^{1/\alpha}$. Condition 2) satisfies the transversality condition of the Hopf bifurcation of system (4). Hence, a Hopf bifurcation occurs at the zero solution when $\mu = \mu_j$. ∎

C. Fractional-Order Model of Dual Congestion Control Algorithms

The dual algorithm is one of important congestion control algorithms in networks, which holds dynamics at links, but static functions at sources. This algorithm can achieve very high utilization but is restricted to a specific class of utility functions [35]. To facilitate a control theoretic study, the congestion control algorithms are often converted into delayed integer-order differential equations [1], [2], [5], [36].

Raina [19] introduced the following dynamical representation of dual congestion control algorithms with a single link and a single delay:

$$\frac{d}{dt}p(t) = \kappa p^m(t)(x(t-\tau) - C) \tag{8}$$

where the variable p is the price at the link, τ is the communication delay, $\kappa > 0$ is the gain parameter, and the scalar $C > 0$ is the capacity of the bottleneck link. In addition, $x(t) = \mathcal{D}(p(t))$ with $\mathcal{D}(p)$, $p \geq 0$, is a nonnegative continuous, strictly decreasing demand function. If $m = 0$, then this is called the delay dual and a possible form of the demand function was identified in [37] as

$$\text{Exponential law}: \quad \mathcal{D}(p) = D_{\max} e^{-\gamma_s p/\tau}$$

where $\gamma_s > 0$ is chosen to ensure local stability and $D_{\max} > C > 0$ is a maximum demand parameter. The fair dual corresponds to

$$m = 1, \quad \mathcal{D}(p) = (w/p)^{1/\gamma}$$

where $w > 0$ may be viewed as a willingness to pay parameter of the user and $\gamma > 0$ is the fair allocation parameter [38].

The integer-order model (8) of dual congestion control algorithms has been extensively studied regarding its stability, bifurcation, and control in the past years. In [19], the local Hopf bifurcation of model (8) was considered by choosing the parameter κ as the bifurcation parameter. The explicit conditions were derived to ensure the onset of stable limit cycles as model (8) just loses its local stability, and the direction of Hopf bifurcations was determined by applying the normal form theory and center manifold theorem. In [21], by selecting the delay τ as the bifurcation parameter, it was demonstrated that the fair dual model (8) loses its stability and a Hopf bifurcation occurs when the delay τ passes through critical values. Moreover, the bifurcating periodic solution was calculated by means of the perturbation method. In [22], the hybrid control was applied to realize the control of the undesirable Hopf bifurcation of model (8).

In this note, we substitute the fractional-order Caputo derivative (1) for the usual integer-order derivative in model (8) to obtain the following delayed fractional-order model of dual congestion control algorithms:

$${}_0^C D_t^\alpha p(t) = \kappa p^m(t)(x(t-\tau) - C) \tag{9}$$

where $\alpha \in (0, 1]$.

III. Stability and Bifurcation

Suppose that p^* is a nonzero equilibrium of (9). Then, it satisfies the following equation:

$$\mathcal{D}(p^*) = C. \tag{10}$$

Remark 7: It should be underlined that (10) does not depend on the fractional order $q \in (0, 1)$. Due to the properties of the Caputo fractional-order derivative, it has been concluded in [39] and [40] that p^* is an equilibrium of the fractional-order system (9) with fractional order $q \in (0, 1)$ if and only if it is an equilibrium of the integer-order system (8). Thus, the same results hold for the existence, uniqueness or multiplicity of equilibria of fractional-order systems, as in the case of integer-order systems. However, this fact of equilibria is no longer true for the initialization issues of pseudostate space representations in fractional-order systems [41], [42].

In the following, the stability and bifurcation properties of the delayed fractional-order model (9) will be investigated in the case of two special dual congestion control algorithms: the delay dual algorithm and the fair dual algorithm.

A. Case of the Delay Dual Algorithm

The fractional-order system (9) of delay dual congestion control algorithms [19], [37] is described by

$${}_0^C D_t^\alpha p(t) = \kappa(x(t-\tau) - C) \tag{11}$$

where $x(t-\tau) = D_{\max} e^{-\gamma_s p(t-\tau)/\tau}$. Let $u(t) = p(t) - p^*$, where $p^* = -(\tau/\gamma_s)\ln(C/D_{\max})$, and shift the equilibrium p^* to the origin. The linearized system of (11) is given by

$${}_0^C D_t^\alpha u(t) = -(\kappa C\gamma_s/\tau)u(t-\tau) \tag{12}$$

with the characteristic equation

$$s^\alpha + (\kappa C\gamma_s/\tau)e^{-s\tau} = 0. \tag{13}$$

Theorem 4: If $(\kappa C\gamma_s/\tau)^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$, where $j \in \mathbb{Z}$, then the equilibrium p^* of system (11) is Lyapunov locally asymptotically stable.

Proof: Assume that $s = \mathrm{i}\omega = \omega(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2})(\omega > 0)$ is a root of (13). Then, one obtains

$$\omega^\alpha \cos(\alpha\pi/2) + (\kappa C\gamma_s/\tau)\cos\omega\tau = 0 \tag{14a}$$

$$\omega^\alpha \sin(\alpha\pi/2) - (\kappa C\gamma_s/\tau)\sin\omega\tau = 0. \tag{14b}$$

Taking square on the both the sides of (14a), (14b) and summing them up gives

$$(\omega^\alpha)^2 + (\kappa C\gamma_s/\tau)^2 + 2\omega^\alpha(\kappa C\gamma_s/\tau)\cos((\alpha\pi/2) + \omega\tau) = 0. \tag{15}$$

Notice that $\kappa C\gamma_s/\tau > 0$. It is easy to see that

$$\begin{aligned} &(\omega^\alpha)^2 + (\kappa C\gamma_s/\tau)^2 + 2\omega^\alpha(\kappa C\gamma_s/\tau)\cos((\alpha\pi/2) + \omega\tau) \\ \geq\ &(\omega^\alpha)^2 + (\kappa C\gamma_s/\tau)^2 - 2\omega^\alpha(\kappa C\gamma_s/\tau) \\ =\ &(\omega^\alpha - \kappa C\gamma_s/\tau)^2. \end{aligned}$$

Obviously, if $(\kappa C\gamma_s/\tau)^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$, then (15) has no positive real roots, meaning that (13) has no purely imaginary roots with positive imaginary parts.

Similarly, it can be proved that (13) has no purely imaginary roots with negative imaginary parts under the assumption $(\kappa C\gamma_s/\tau)^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$.

Therefore, if $(\kappa C\gamma_s/\tau)^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$, then the characteristic equation (13) has no purely imaginary roots.

Moreover, it can be seen that the coefficient matrix A of the linearized model (12) has one eigenvalue $\lambda = -\kappa C\gamma_s/\tau < 0$ satisfying $|\arg(\lambda)| > \alpha\pi/2$. Applying Remark 6, the equilibrium p^* of (11) is Lyapunov asymptotically stable, thus completing the proof. ■

Remark 8: The stability of congestion control algorithms has been studied based on integer-order dynamical systems in [1]–[4], [38]. However, to the best of the authors' knowledge, the theoretical results on the stability of fractional-order dynamical systems of congestion control algorithms have not been reported yet.

Lemma 1: If $\kappa = \kappa_j, j = 0, 1, \ldots$, then (13) has a pair of purely imaginary roots $\pm \mathrm{i}\omega_0$, where

$$\kappa_j = [(2j+1)\pi - \alpha\pi/2]^\alpha \tau^{1-\alpha}/(C\gamma_s) \tag{16a}$$

$$\omega_0 = (\kappa_j C\gamma_s/\tau)^{1/\alpha}. \tag{16b}$$

Proof: It can be seen from the proof of Theorem 4 that (13) has a pair of purely imaginary roots when $(\kappa C\gamma_s/\tau)^{1/\alpha} = \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$. Therefore, the conclusion follows immediately. ■

Remark 9: Lemma 1 illustrates that the proposed condition 1) in Theorem 3 is reached for the delayed fractional-order system (11) of delay dual congestion control algorithms when (16a), (16b) holds.

Lemma 2: Let $s(\kappa) = \rho(\kappa) + \mathrm{i}\omega(\kappa)$ be the root of (13) satisfying $\rho(\kappa_j) = 0$ and $\omega(\kappa_j) = \omega_0 > 0, j = 0, 1, \ldots$. Then,

$$\left.\frac{d\rho}{d\kappa}\right|_{\kappa=\kappa_j} > 0.$$

Proof: Substituting $s(\kappa)$ into (13) and differentiating both the sides of the resulting equation with respect to κ yields

$$\alpha s^{\alpha-1}\frac{ds}{d\kappa} + C\frac{\gamma_s}{\tau}e^{-s\tau}\left(1 - \kappa\tau\frac{ds}{d\kappa}\right) = 0.$$

Thus,

$$\frac{ds}{d\kappa} = \frac{C\gamma_s e^{-s\tau}}{C\gamma_s\tau\kappa e^{-s\tau} - \alpha\tau s^{\alpha-1}}.$$

Note that $s(\kappa) = \rho(\kappa) + \mathrm{i}\omega(\kappa) = r(\cos\theta + \mathrm{i}\sin\theta)$ is the root of (13). One has

$$\frac{ds}{d\kappa} = \frac{C\gamma_s e^{-\rho\tau}[\cos(\omega\tau) - \mathrm{i}\sin(\omega\tau)]}{C\gamma_s\tau\kappa e^{-\rho\tau}[\cos(\omega\tau) - \mathrm{i}\sin(\omega\tau)] - \alpha\tau[\rho + \mathrm{i}\omega]^{\alpha-1}}.$$

From this one obtains

$$\frac{d\rho}{d\kappa} = C\gamma_s e^{-\rho\tau}\frac{M(\kappa)\cos\omega\tau + N(\kappa)\sin\omega\tau}{M^2(\kappa) + N^2(\kappa)}$$

in which

$$M(\kappa) = C\gamma_s\tau\kappa e^{-\rho\tau}\cos\omega\tau - \alpha\tau r^{\alpha-1}\cos(\alpha-1)\theta$$
$$N(\kappa) = C\gamma_s\tau\kappa e^{-\rho\tau}\sin\omega\tau + \alpha\tau r^{\alpha-1}\sin(\alpha-1)\theta.$$

Replacing κ by κ_j, it follows that

$$\left.\frac{d\rho}{d\kappa}\right|_{\kappa=\kappa_j} = C\gamma_s\frac{M(\kappa_j)\cos\omega_0\tau + N(\kappa_j)\sin\omega_0\tau}{M^2(\kappa_j) + N^2(\kappa_j)}$$
$$= C\gamma_s\frac{C\gamma_s\tau\kappa_j - \alpha\tau\omega_0^{\alpha-1}\cos[\omega_0\tau + (\alpha-1)\frac{\pi}{2}]}{M^2(\kappa_j) + N^2(\kappa_j)}$$

where

$$M(\kappa_j) = C\gamma_s\tau\kappa_j\cos\omega_0\tau - \alpha\tau\omega_0^{\alpha-1}\cos((\alpha-1)\pi/2)$$
$$N(\kappa_j) = C\gamma_s\tau\kappa_j\sin\omega_0\tau + \alpha\tau\omega_0^{\alpha-1}\sin((\alpha-1)\pi/2).$$

It can be seen from (16a), (16b) that $\omega_0\tau = (2j+1)\pi - \alpha\pi/2$, implying that $\cos[\omega_0\tau + (\alpha-1)\pi/2] = 0$. Therefore,

$$\left.\frac{d\rho}{d\kappa}\right|_{\kappa=\kappa_j} = \frac{C^2\gamma_s^2\tau\kappa_j}{M^2(\kappa_j) + N^2(\kappa_j)} > 0.$$

The conclusion follows. ■

Remark 10: Lemma 2 indicates that the transversality condition 2) in Theorem 3 is satisfied for the delayed fractional-order system (11) of delay dual congestion control algorithms.

Theorem 5: For system (11), the following results hold.

1) The equilibrium p^* of system (11) is asymptotically stable for $\kappa \in (0, \kappa_0)$, and unstable when $\kappa > \kappa_0$.
2) System (11) undergoes a Hopf bifurcation at the equilibrium p^* when $\kappa = \kappa_j, j = 0, 1, \ldots$. Here κ_j is defined as in (16a), (16b).

Proof: From (16b), (16b) it is known that $0 < \kappa_0 < \kappa_1 < \kappa_2 < \ldots$.

1) The definition of κ_0 implies that the condition $(\kappa C\gamma_s/\tau)^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$ stated in Theorem 4 is satisfied when $\kappa \in (0, \kappa_0)$. Thus, system (11) is asymptotically stable for $\kappa \in (0, \kappa_0)$. On the other hand, from Lemma 1, the characteristic equation (13) has a pair of purely imaginary roots when $\kappa = \kappa_0$. Together with Lemma 2, (13) has at least a root with positive real part when $\kappa > \kappa_0$. Thus, the conclusion follows.
2) From Remarks 9 and 10, we know that the proposed conditions 1) and 2) for the occurrence of Hopf bifurcations in Theorem 3 are satisfied for the fractional-order system (11). Hence, a Hopf bifurcation occurs at the equilibrium p^* when $\kappa = \kappa_j$.

■

Remark 11: The Hopf bifurcation theory of fractional-order dynamical systems is still an open problem. In [23] and [24], the conditions for the occurrence of Hopf bifurcations were constructed for fractional-order systems without delays based on the observations from numerical simulations. However, there are few theoretical results on the Hopf bifurcation of delayed fractional-order systems. Theorem 5 gives sufficient conditions for the occurrence of Hopf bifurcations in delayed fractional-order congestion control systems for the first time.

Example 1: Let the link capacity be 1.25 Mb/s and the time unit be 40 ms. If the packet sizes are 1000 bytes each, then the link capacity can be expressed as $C = 50$ packets per time unit [21]. In addition, we set $D_{\max} = 70, \gamma_s = 2$, and the communication delay τ is 3 ms. The nonzero equilibrium is $p^* = 0.5047$. For system (11) with $\alpha = 0.86$, it follows from (16a), (16b) that

$$\kappa_0 = 0.0192, \quad \omega_0 = 0.5969.$$

From Theorems 4 and 5, it is known that when $\kappa \in (0, \kappa_0)$, the trajectories converge to the equilibrium p^*, as shown in Fig. 1, while with κ being increased to pass through κ_0, the equilibrium p^* loses its stability and a Hopf bifurcation occurs, as shown in Fig. 2. The numerical solution is derived by using the Adams–Bashforth–Moulton predictor–corrector method [43].

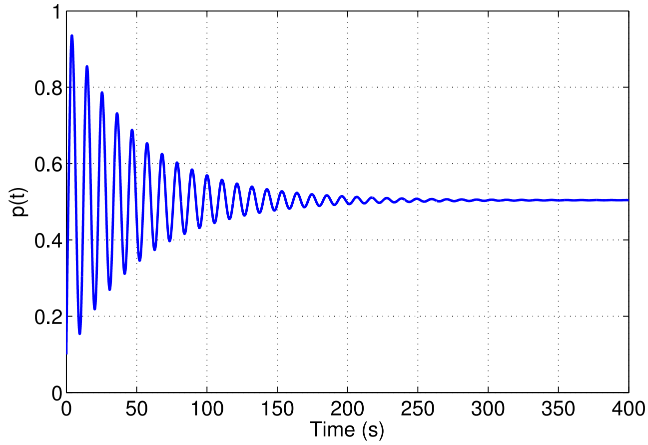

Fig. 1. Equilibrium $p^* = 0.5047$ of system (11) with $\alpha = 0.86$ is asymptotically stable, where $D_{\max} = 70, C = 50, \gamma_s = 2, \tau = 3$, and $\kappa = 0.018 < \kappa_0 = 0.0192$.

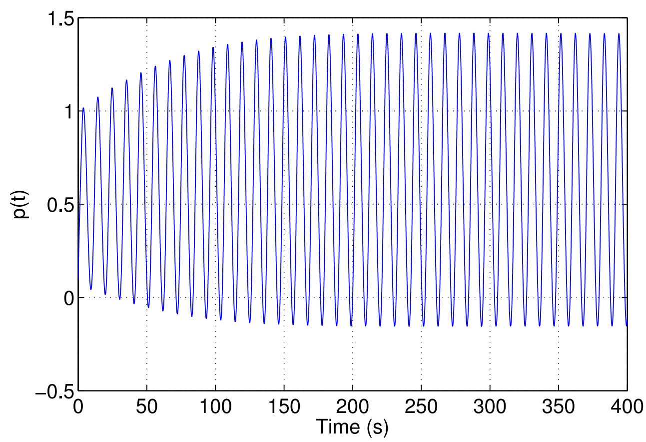

Fig. 2. Periodic oscillation bifurcates from the equilibrium $p^* = 0.5047$ of system (11) with $\alpha = 0.86$, where $D_{\max} = 70, C = 50, \gamma_s = 2, \tau = 3$, and $\kappa = 0.02 > \kappa_0 = 0.0192$.

Theorem 6: For the bifurcation curve $\kappa_0(\alpha) = (\pi - \alpha\pi/2)^\alpha \tau^{1-\alpha}/(C\gamma_s)$ in (16a), (16b), the following results hold.

1) If $\tau \geq \pi$, then $\kappa_0(\alpha)$ is a monotonically decreasing function in $(0, 1]$.
2) If $0 < \tau \leq \pi/(2e)$, then $\kappa_0(\alpha)$ is a monotonically increasing function in $(0, 1]$.
3) If $\pi/(2e) < \tau < \pi$, then $\kappa_0(\alpha)$ is monotonically increasing when $\alpha \in (0, \alpha^*)$, and monotonically decreasing when $\alpha \in (\alpha^*, 1]$, where α^* is the unique positive root of $h(\alpha) = \ln \tau$.

Proof: It follows from the definition of $\kappa_0(\alpha)$ that

$$\kappa_0'(\alpha) = \frac{(\pi - \frac{\alpha\pi}{2})^\alpha \tau^{1-\alpha}}{C\gamma_s}[h(\alpha) - \ln \tau]$$

where $h(\alpha) = \ln(\pi - \alpha\pi/2) - (\alpha\pi/2)/(\pi - \alpha\pi/2)$. It is easy to prove that

$$h'(\alpha) = -\frac{\pi/2}{\pi - \alpha\pi/2} - \frac{\pi^2/2}{(\pi - \alpha\pi/2)^2} < 0$$

which indicates that $h(\alpha)$ is a monotonically decreasing function in $(0, 1]$. Thus, we have $\ln(\pi/2) - 1 = h(1) \leq h(\alpha) < h(0) = \ln \pi$.

1) If $\tau \geq \pi$, then $h(\alpha) - \ln \tau < \ln \pi - \ln \tau \leq 0$ for $\alpha \in (0, 1]$. Note that $C > 0, \gamma_s > 0$, and $(\pi - \alpha\pi/2)^\alpha > 0$. Therefore, we have $\kappa_0'(\alpha) < 0$, implying that $\kappa_0(\alpha)$ is a monotonically decreasing function in $(0, 1]$.
2) If $0 < \tau \leq \pi/(2e)$, then $h(\alpha) - \ln \tau \geq \ln(\pi/2) - 1 - \ln \tau \geq 0$. Thus, we have $\kappa_0'(\alpha) \geq 0$ for $\alpha \in (0, 1]$. This implies that $\kappa_0(\alpha)$ is a monotonically increasing function in $(0, 1]$.
3) If $\pi/(2e) < \tau < \pi$, then $h(\alpha) - \ln \tau > 0$ for $\alpha \in (0, \alpha^*)$, and $h(\alpha) - \ln \tau < 0$ when $\alpha \in (\alpha^*, 1]$. This implies that $\kappa_0'(\alpha) > 0$ for $\alpha \in (0, \alpha^*)$, and $\kappa_0'(\alpha) < 0$ for $\alpha \in (\alpha^*, 1]$. Hence, the proof is complete.

■

The effect of the order and delay variation on the response of system (11) is illustrated in Figs. 3 and 4. Fig. 3 shows the bifurcation curves of system (11) in the $\alpha - \kappa_0$ plane. With increasing the delay τ, the critical value κ_0 increases for a fixed order α, that is to say, the onset of the Hopf bifurcation is postponed. Thus, the stability domain is extended, and system (11) possesses a stable price at the link in a larger delay. Fig. 4 displays the bifurcation surface of system (11) in the (α, τ, κ_0) space.

B. Case of the Fair Dual Algorithm

The fractional-order congestion control system (9) of fair dual algorithms [19], [38] is described by

$$ {}_0^C D_t^\alpha p(t) = \kappa p(t)(x(t-\tau) - C) \tag{17}$$

where $x(t-\tau) = (w/p(t-\tau))^{1/\gamma}$. Let $u(t) = p(t) - p^*$, where $p^* = w/C^\gamma$. The linearized system of (17) is given by

$$ {}_0^C D_t^\alpha u(t) = -(\kappa C/\gamma) u(t-\tau)$$

with the characteristic equation

$$s^\alpha + (\kappa C/\gamma)e^{-s\tau} = 0. \tag{18}$$

Theorem 7: If $(\kappa C/\gamma)^{1/\alpha} \neq \frac{1}{\tau}[(2j+1)\pi - \alpha\pi/2]$, where $j \in \mathbb{Z}$, then the equilibrium p^* of system (17) is Lyapunov asymptotically stable.

The proof of Theorem 7 is similar to the proof of Theorem 4. From Theorem 7, it is straightforward to obtain the following result.

Lemma 3: If $\kappa = \kappa_j, j = 0, 1, \ldots$, then (18) has a pair of purely imaginary roots $\pm i\omega_0$, where

$$\kappa_j = [(2j+1)\pi - \alpha\pi/2]^\alpha \gamma/(C\tau^\alpha) \tag{19a}$$

$$\omega_0 = (\kappa_j C/\gamma)^{1/\alpha}. \tag{19b}$$

Lemma 4: Let $s(\kappa) = \rho(\kappa) + i\omega(\kappa)$ be the root of (18) satisfying $\rho(\kappa_j) = 0$ and $\omega(\kappa_j) = \omega_0 > 0, j = 0, 1, \ldots$. Then,

$$\left.\frac{d\rho}{d\kappa}\right|_{\kappa=\kappa_j} = \frac{C^2\tau\kappa_j}{M^2(\kappa_j) + N^2(\kappa_j)} > 0$$

where

$$M(\kappa_j) = C\tau\kappa_j \cos\omega_0\tau - \alpha\gamma\omega_0^{\alpha-1}\cos((\alpha-1)\pi/2)$$

$$N(\kappa_j) = C\tau\kappa_j \sin\omega_0\tau + \alpha\gamma\omega_0^{\alpha-1}\sin((\alpha-1)\pi/2).$$

The proof of Lemma 4 is similar to the proof of Lemma 2.

Theorem 8: For system (17), the following results hold.

1) The equilibrium p^* of system (17) is asymptotically stable for $\kappa \in (0, \kappa_0)$, and unstable when $\kappa > \kappa_0$.
2) System (17) undergoes a Hopf bifurcation at the equilibrium p^* when $\kappa = \kappa_j, j = 0, 1, \ldots$. Here κ_j is defined as in (19a), (19b).

The proof of Theorem 8 is similar to that of Theorem 5.

Note that the appearance of γ can dictate the onset $\kappa_0(\alpha, \tau) = (\pi - \alpha\pi/2)^\alpha\gamma/(C\tau^\alpha)$ of the Hopf bifurcation. We obtain the TCP

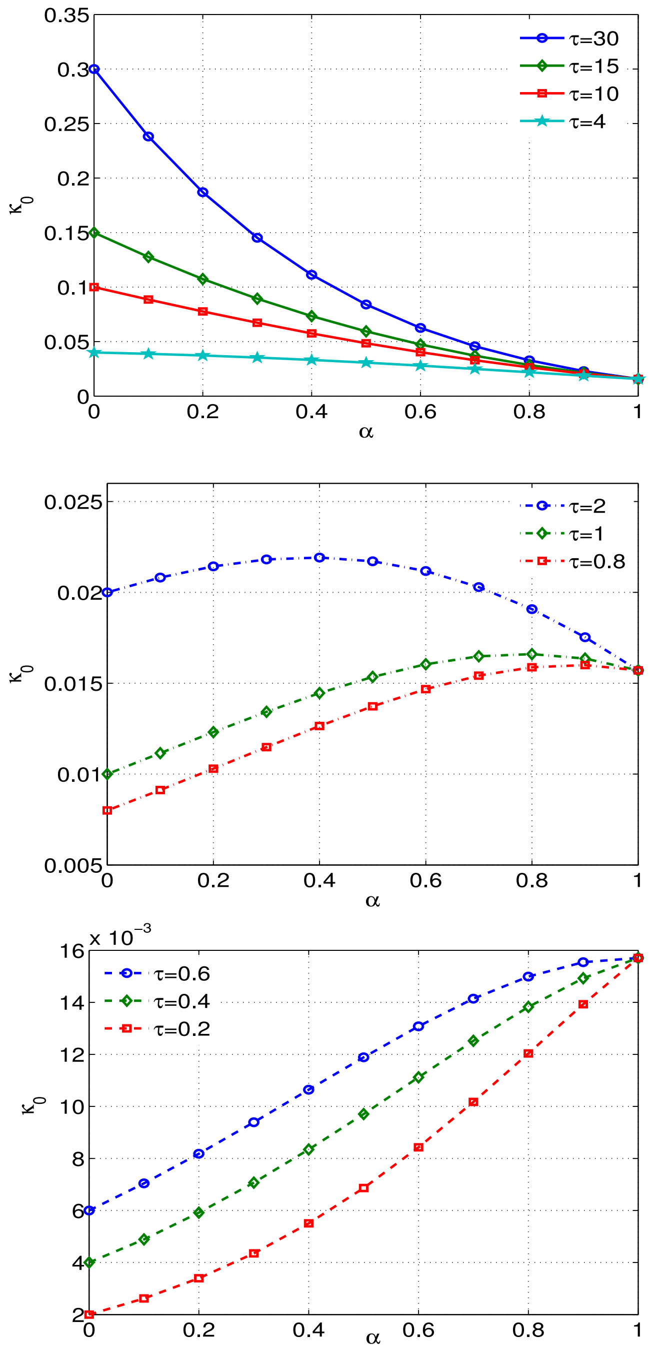

Fig. 3. Bifurcation curves $\kappa_0(\alpha) = (\pi - \alpha\pi/2)^\alpha \tau^{1-\alpha}/(C\gamma_s)$ of system (11) with $C = 50, \gamma_s = 2$, and different values of τ: τ, respectively, equals 30, 15, 10, 4, 2, 1, 0.8, 0.6, 0.4, and 0.2.

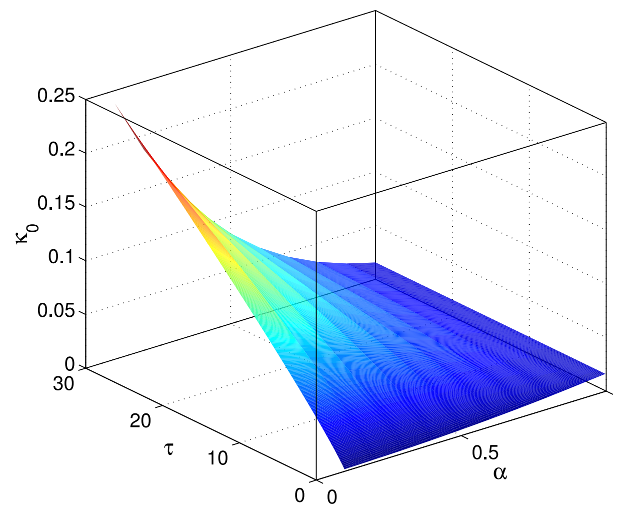

Fig. 4. Bifurcation surface $\kappa_0(\alpha,\tau) = (\pi - \alpha\pi/2)^\alpha \tau^{1-\alpha}/(C\gamma_s)$ of system (11) with $C = 50, \gamma_s = 2$.

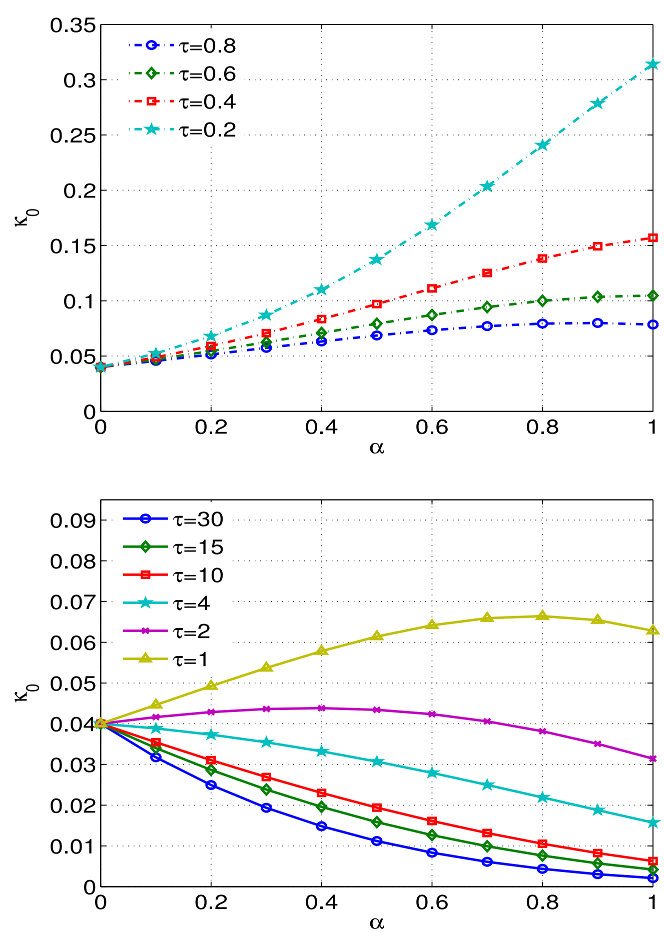

Fig. 5. Bifurcation curves $\kappa_0(\alpha,\tau) = (\pi - \alpha\pi/2)^\alpha \gamma/(C\tau^\alpha)$ of system (17) with $C = 50, \gamma = 2$, and different values of τ: τ, respectively, equals 30, 15, 10, 4, 2, 1, 0.8, 0.6, 0.4, and 0.2.

fairness [38] with $\gamma = 2$ and the proportional fairness [21] with $\gamma = 1$. Therefore, the TCP fair algorithm will have a Hopf bifurcation with twice the onset of its proportionally fair counterpart. The effect of the order and delay variation on the response of system (17) is illustrated in Figs. 5–7. Figs. 5 and 6 show the bifurcation curves of system (17) in the $\alpha - \kappa_0$ plane for the TCP fairness and the proportional fairness, respectively. For a fixed order α, decreasing τ postpones the onset of the Hopf bifurcation and reduces the instability. Fig. 7 displays the bifurcation surfaces of system (17) for the TCP fairness and the proportional fairness.

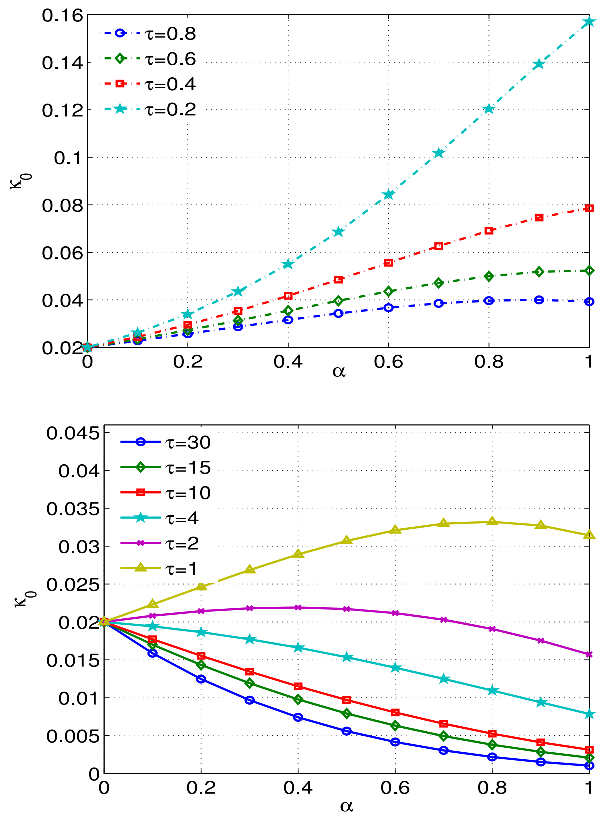

Fig. 6. Bifurcation curves $\kappa_0(\alpha, \tau) = (\pi - \alpha\pi/2)^\alpha \gamma/(C\tau^\alpha)$ of system (17) with $C = 50$, $\gamma = 1$, and different values of τ: τ, respectively, equals 30, 15, 10, 4, 2, 1, 0.8, 0.6, 0.4, and 0.2.

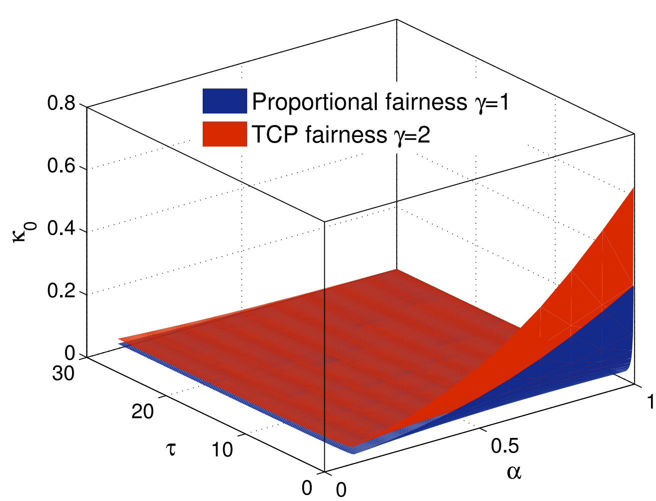

Fig. 7. Bifurcation surface $\kappa_0(\alpha, \tau) = (\pi - \alpha\pi/2)^\alpha \gamma/(C\tau^\alpha)$ of system (17) with $C = 50$ and different values of γ: γ, respectively, equals 2 and 1.

IV. Conclusion

There have been many results on dynamical characteristics for a variety of integer-order congestion control systems over the past decades. However, the study of dynamics for more accurate fractional-order congestion systems can be more significant. In this note, we have extended a delayed integer-order model of dual congestion control algorithms to a delayed fractional-order counterpart. The stability and bifurcation properties have been investigated for the delayed fractional-order system in the case of two special dual congestion control algorithms: the delay dual algorithm and the fair dual algorithm. The stability conditions for the delayed fractional-order congestion system have been established based on the stability theorem on delayed fractional-order differential equations. We have proposed the conditions for the occurrence of Hopf bifurcations for general delayed fractional-order systems. Our delayed fractional-order system can exhibit a Hopf-type bifurcation (i.e., periodic oscillations appear) as the gain parameter passes through the critical values which can be determined exactly. Bifurcation curves and bifurcation surfaces have been displayed. Moreover, it has been observed that the critical values of Hopf bifurcations are sensitive to the change of the order and delay. These observations allow us to design the Hopf bifurcation of congestion control systems with the desired bifurcation point by adjusting the order and delay.

Although the Caputo derivative has its own limitation in describing the characteristics of dynamical systems, it has been demonstrated in recent years that such a derivative is a useful tool for modeling many physical systems [44]. For some fractional-order systems, the "state" of the system is not given by the dynamic variable vector, because the initialization vector, traditionally a vector of constants, has been shown to be time varying [26], [28]. Our future work will focus on the Hopf bifurcation analysis for initialized fractional-order dual congestion control algorithms.

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