A State-Observer-Based Approach for Synchronization in Complex Dynamical Networks

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***Abstract*—In this paper, a new approach for synchronization of complex dynamical networks is proposed based on state observer design. Unlike the common diagonally coupling networks, where full state coupling is typically needed between two nodes, here it is suggested that only a scalar coupling signal is required to achieve network synchronization. Some conditions for synchronization, in the form of an inequality, are established based on the Lyapunov stability theory, which can be transformed to a linear matrix inequality and easily solved by a numerical toolbox. Two typical dynamical network configurations, i.e., global coupling and nearest-neighbor coupling, with each node being a modified Chua's circuit, are simulated. It is demonstrated that the proposed scheme is effective in achieving the expected chaos synchronization in the complex network.**

***Index Terms*—Complex dynamical network, linear matrix inequality (LMI), Lyapunov stability, state observer, synchronization.**

I. INTRODUCTION

RECENTLY, there has been increasing interest in the study of complex dynamical networks and their collective behaviors in synchronization [1]–[11], [16]–[22]. Different mathematical models have been proposed in order to describe various complex dynamical networks in the real world, such as the Erdös–Rényi (E-R) random-graph models [34], small-world models [5], scale-free models [1], [4], and so on. The E-R model is one of the oldest and perhaps also the most rigorous mathematical platforms for studying statistical network behaviors, but it cannot capture some typical dynamical and statistical phenomena in many real-life large-scale complex networks, such as large clustering with small average distance and power-law degree distribution. Hence, small-world models, scale-free models, and their variants as well as evolving network models have recently been developed to represent various real-world networks, including the Internet, the World-Wide-Web, power grids, metabolic networks, social relationship networks, and so on.

Manuscript received October 8, 2005; revised May 16, 2006. This work was supported in part by the "Qing Lan Project" of Jiangsu Province, China, under Contract JS200407, the Key Project of Natural Science Foundation of Universities of Jiangsu Province, China, under Contract 04KJA510092, the "Climbing Project" of NJUPT, China, under Contract NY206026, and the Research Grants Council of Hong Kong SAR, China, under Grant CityU1157/03E.

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Digital Object Identifier 10.1109/TCSI.2006.883876

In the real world, synchronizing a group of dynamical nodes in a complex network topology, as one of the basic collective behaviors of a dynamical network, is important and yet quite challenging [3], [8]–[11], [14], [16]–[22]. Conditions for synchronization are critical for many engineering applications, such as secure communications [25] and harmonic oscillation generation, to name just a couple. Similar studies can also be found in language emergence and development, for which a common set of vocabularies is to be found, or in organization management, where the efficiency of agents can be improved via behavioral synchronization [16]. Various networks have been studied, including regular networks [10], [11], [14] and small-world and scale-free networks [3], [8], [9], [16]–[22], in addition to a large amount of work on the classic random-graph networks.

Most of the underlying models [16]–[22] assume diagonal coupling among the nodes in a network, implying that all of the state variables of a node have to be transmitted to its connected neighbors. This assumption not only leads to a very dense network topology or a large capacity of connection channels, but also is impractical for real engineering network design and implementation, such as communication networks, in which having too many links or too wide bandwidths in communication channels among the users is very unlikely.

This paper addresses the question of whether it is possible to have synchronization achieved in a network with nodes connected only through one-dimensional links. A positive answer is given and, based on the state observer approach, a general synchronization scheme for such a network topology is proposed and studied. The state observer approach has been applied to chaos synchronization between two chaotic circuits or systems [26]–[32], where only one scalar driving signal is used. Here, the technique is extended to a large-scale complex network, where multiple nodes are to be synchronized. Based on the Lyapunov stability theory and the linear matrix inequality (LMI) technique [33], some criteria are established in the form of LMIs, which can be easily solved by an existing LMI toolbox.

The remainder of this paper is organized as follows. In Section II, a dynamical network model based on the state observer approach is proposed. Synchronization of such a dynamical network and some criteria are then presented in Section III. In Section IV, two typical dynamical networks, with global coupling and nearest-neighbor coupling, respectively, where each node is a modified Chua's circuit [23], [24], are simulated, illustrating the effectiveness of the theoretical results and validating the criteria derived in the paper. Finally, some concluding remarks are provided in Section V.

II. Dynamical Network Model

Consider a dynamical network consisting of N linearly and diffusively coupled identical nodes, with each node being an n-dimensional dynamical system. Based on the state observer approach, the proposed dynamical network model is described by

$$\dot{\mathbf{x}}_i = \mathrm{f}(\mathbf{x}_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}\mathbf{L}(y_j - y_i), \qquad i = 1, 2, \cdots, N \tag{1}$$

where $\mathbf{x}_i = [x_{i1}\ x_{i2}\ \cdots\ x_{\mathrm{in}}]^T \in \mathbf{R}^n$ is the state variable of node i, $y_i \in R, i = 1, 2, \cdots, N$, is the output variable (scalar) of node i (a dynamical system), $\mathbf{L} = [l_1\ l_2\ \cdots\ l_n]^T$ is the observer gain matrix to be designed in order to achieve synchronization, and $\mathbf{C} = (c_{ij})_{N\times N}$ is the coupling configuration matrix representing the coupling strength and the topological structure of the network, in which c_{ij} is defined as follows. If there is a connection between node i and node $j(j \neq i)$, then $c_{ij} = c_{ji} > 0$, otherwise, $c_{ij} = c_{ji} = 0$ $(j \neq i)$, and the diagonal elements of matrix $\mathbf{C}$ are defined by

$$c_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}, \qquad i = 1, 2, \cdots, N. \tag{2}$$

Assuming that there is no isolate cluster in the network, the coupling configuration matrix $\mathbf{C} = (c_{ij})_{N\times N}$ will be symmetrical and irreducible.

From (1) and (2), dynamical network (1) can be rewritten as

$$\dot{\mathbf{x}}_i = \mathrm{f}(\mathbf{x}_i) + \sum_{j=1}^{N} c_{ij}\mathbf{L}y_j, \qquad i = 1, 2, \cdots, N. \tag{3}$$

Remark 1: From (1) or (3), one can see that only one scalar signal is needed for coupling between two directly connected nodes in the network, while the other network synchronization methods generally require n state variables for coupling between any two directly connected nodes [16]–[22].

Let

$$y_i = \mathbf{H}\mathbf{x}_i, \qquad i = 1, 2, \cdots, N \tag{4}$$

where $\mathbf{H} = [h_1\ h_2\ \cdots\ h_n]$ is the observer matrix. Thus, y_i is a linear combination of the state variables of node i. Substituting (4) into (3) gives

$$\dot{\mathbf{x}}_i = \mathrm{f}(\mathbf{x}_i) + \sum_{j=1}^{N} c_{ij}\mathbf{L}\mathbf{H}\mathbf{x}_j. \tag{5}$$

III. Network Synchronization and its Criteria

The dynamical network (5) is said to achieve (asymptotical) synchronization [11] if

$$\lim_{t\to\infty} \|\mathbf{x}_i(t) - \mathbf{s}(t)\| = 0, \qquad i = 1, 2, \cdots, N \tag{6}$$

where $\|\cdot\|$ is the Euclidean norm and $\mathbf{s}(t) \in \mathbf{R}^n$ satisfies

$$\dot{\mathbf{s}}(t) = \mathrm{f}\left(\mathbf{s}(t)\right). \tag{7}$$

Here, $\mathbf{s}(t)$ can be an equilibrium point, a periodic orbit, or even a chaotic orbit. It is obvious that the stability of the synchronized states (6) of network (5) is determined by the dynamics of the isolate node (7), the coupling matrix $\mathbf{C}$, the observer gain matrix $\mathbf{L}$, and the observer matrix $\mathbf{H}$.

Assuming $\mathbf{e}_i = \mathbf{x}_i(t) - \mathbf{s}(t)$ and linearizing the network (5) about $\mathbf{s}(t)$, one has

$$\dot{\mathbf{e}}_i = D\mathrm{f}\left(\mathbf{s}(t)\right)\mathbf{e}_i + \sum_{j=1}^{N} c_{ij}\mathbf{L}\mathbf{H}\mathbf{e}_j \tag{8}$$

where $D\mathrm{f}(\mathbf{s}(t))$ is the Jacobian of the function $\mathrm{f}(.)$ at $\mathbf{s}(t)$.

It follows from (8), that (9), which is shown at the bottom of this page, is true.

Based on the system stability theory, if (9) is asymptotically stable about zero, the dynamical network (3) or (5) will be asymptotically synchronized.

In order to derive some criteria for choosing $\mathbf{L}$ and $\mathbf{H}$ to ensure synchronization in network (5), a special feature of matrix $\mathbf{C}$ is firstly needed, given as follows.

Lemma 1 [17], [22]: One eigenvalue of matrix $\mathbf{C} = (c_{ij})_{N\times N}$ is zero, with multiplicity 1, and the other eigenvalues of $\mathbf{C}$ are strictly negative.

Theorem 1: Consider the dynamical network (5) and assume that the Jacobian matrix $D\mathrm{f}(\mathbf{s}(t))$ is exponentially stable. If the following $N-1$ n-dimensional linear time-varying systems are exponentially stable:

$$\dot{\mathbf{w}}_k = \left(D\mathrm{f}\left(\mathbf{s}(t)\right) + \lambda_k\mathbf{L}\mathbf{H}\right)\mathbf{w}_k, \qquad k = 2, \cdots, N \tag{10}$$

then the dynamical network (5) is exponentially synchronized.

Proof: From (8), one has

$$\dot{\mathbf{e}}_i = D\mathrm{f}\left(\mathbf{s}(t)\right)\mathbf{e}_i + \mathbf{L}\mathbf{H}[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_N]\begin{bmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{iN} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \vdots \\ \dot{\mathbf{e}}_N \end{bmatrix} = \begin{bmatrix} D\mathrm{f}\left(\mathbf{s}(t)\right) + c_{11}\mathbf{L}\mathbf{H} & c_{12}\mathbf{L}\mathbf{H} & \cdots & c_{1N}\mathbf{L}\mathbf{H} \\ c_{21}\mathbf{L}\mathbf{H} & Df\left(\mathbf{s}(t)\right) + c_{22}\mathbf{L}\mathbf{H} & \cdots & c_{2N}\mathbf{L}\mathbf{H} \\ \vdots & \vdots & \cdots & \vdots \\ c_{N1}\mathbf{L}\mathbf{H} & c_{N2}\mathbf{L}\mathbf{H} & \cdots & Df\left(\mathbf{s}(t)\right) + c_{NN}\mathbf{L}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_N \end{bmatrix} \tag{9}$$

and

$$[\dot{\mathbf{e}}_1 \quad \dot{\mathbf{e}}_2 \quad \cdots \quad \dot{\mathbf{e}}_N] = D\mathrm{f}\left(\mathbf{s}(t)\right)[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_N] + \mathbf{LH}[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_N]\begin{bmatrix} c_{11} & c_{21} & \cdots & c_{N1} \\ c_{12} & c_{22} & \cdots & c_{N2} \\ \vdots & \vdots & \cdots & \vdots \\ c_{1N} & c_{2N} & \cdots & c_{NN} \end{bmatrix}. \quad (12)$$

Letting $\boldsymbol{\eta} = [\mathbf{e}_1 \ \mathbf{e}_2 \ \cdots \ \mathbf{e}_N]$ gives a matrix equation

$$\dot{\boldsymbol{\eta}} = D\mathrm{f}\left(\mathbf{s}(t)\right)\boldsymbol{\eta} + \mathbf{LH}\boldsymbol{\eta} \cdot \mathbf{C}^T. \quad (13)$$

Since $\mathbf{C}$ is a real symmetrical matrix, there exists a unitary matrix $\boldsymbol{\psi} = [\psi_1 \ \psi_2 \ \cdots \ \psi_N]$ such that

$$\mathbf{C}^T\boldsymbol{\psi} = \boldsymbol{\psi}\boldsymbol{\Lambda}, \ \boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N). \quad (14)$$

From (13) and (14), it follows that

$$\begin{aligned} \dot{\boldsymbol{\eta}}\boldsymbol{\psi} &= D\mathrm{f}\left(\mathbf{s}(t)\right)\boldsymbol{\eta}\boldsymbol{\psi} + \mathbf{LH}\boldsymbol{\eta} \cdot \mathbf{C}^T\boldsymbol{\psi} \\ &= D\mathrm{f}\left(\mathbf{s}(t)\right)\boldsymbol{\eta}\boldsymbol{\psi} + \mathbf{LH}\boldsymbol{\eta} \cdot \boldsymbol{\psi}\boldsymbol{\Lambda}. \end{aligned} \quad (15)$$

Let $\mathbf{W}(t) = \boldsymbol{\eta}\boldsymbol{\psi}$ and $\mathbf{W} = [\mathbf{w}_1 \ \mathbf{w}_2 \ \cdots \ \mathbf{w}_N]$. Then

$$\dot{\mathbf{W}} = D\mathrm{f}\left(\mathbf{s}(t)\right)\mathbf{W} + \mathbf{LHW}\boldsymbol{\Lambda} \quad (16)$$

and

$$\dot{\mathbf{w}}_k = \left(D\mathrm{f}\left(\mathbf{s}(t)\right) + \lambda_k\mathbf{LH}\right)\mathbf{w}_k, \qquad k = 1, 2, \cdots, N. \quad (17)$$

According to Lemma 1, one eigenvalue of matrix $\mathbf{C}$ is $\lambda_1 = 0$, and the corresponding linear system in (17) is

$$\dot{\mathbf{w}}_1 = D\mathrm{f}\left(\mathbf{s}(t)\right)\mathbf{w}_1 \quad (18)$$

which is the corresponding linearized system of any individual node $\dot{\mathbf{x}} = \mathrm{f}(\mathbf{x})$ at $\mathbf{x} = \mathbf{s}(t)$. For all $k > 1$, system (17) is system (10).

Therefore, if the following $N-1$ n-dimensional linear time-varying systems are exponentially stable:

$$\dot{\mathbf{w}}_k = \left(D\mathrm{f}\left(\mathbf{s}(t)\right) + \lambda_k\mathbf{LH}\right)\mathbf{w}_k, \qquad k = 2, \cdots, N \quad (19)$$

which is the same as (10), then the network (5) is exponentially synchronized. □

Remark 2: If the individual node $\dot{\mathbf{x}} = \mathrm{f}(\mathbf{x})$ is a periodic system or a chaotic system with a bounded attractor, then one can also obtain a similar result (Corollary 1) following the above procedure, according to the definition of chaos synchronization in a dynamical network, based on the concept of transverse errors [19], [21].

Corollary 1: Consider the dynamical network (5) and assume that each isolate node is a periodic system or a chaotic system. If the following $N-1$ n-dimensional linear time-varying systems are exponentially stable:

$$\dot{\mathbf{w}}_k = \left(D\mathrm{f}\left(\mathbf{s}(t)\right) + \lambda_k\mathbf{LH}\right)\mathbf{w}_k, \qquad k = 2, \cdots, N \quad (20)$$

then the dynamical network with chaotic nodes is exponentially synchronized.

Assume that

$$D\mathrm{f}\left(\mathbf{s}(t)\right) = \mathbf{A} + \mathrm{g}\left(\mathbf{s}(t)\right) \quad (21)$$

where $\mathbf{A} \in \mathbf{R}^{n\times n}$ is a constant matrix, and $\mathrm{g}(\mathbf{s}(t)) \in \mathbf{R}^{n\times n}$ is a continuous nonlinear function satisfying

$$\|\mathrm{g}\left(\mathbf{s}(t)\right)\| \le \rho \quad (22)$$

uniformly in t, where ρ is a constant.

From (20) and (21), one has

$$\dot{\mathbf{w}}_k = \left(\mathbf{A} + \mathrm{g}\left(\mathbf{s}(t)\right) + \lambda_k\mathbf{LH}\right)\mathbf{w}_k, \qquad k = 2, \cdots, N. \quad (23)$$

Theorem 2: Assume that the Jacobian $D\mathrm{f}(\mathbf{s}(t))$ satisfies (21) and (22) and the pair $(\mathbf{A}, \mathbf{H})$ is observable. If a suitable observer gain matrix $\mathbf{L}$ is selected such that

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} + \lambda_k\mathbf{H}^T\mathbf{L}^T\mathbf{P} + \lambda_k\mathbf{PLH} + \rho^2\mathbf{PP} + \mathbf{I} + 2\delta\mathbf{P} < 0, \quad k = 2, \cdots, N \quad (24)$$

where $\mathbf{P}$ is a positive definite and symmetric matrix, $\mathbf{I}$ is the identity matrix, and δ is a positive constant, then the error dynamical system (23) will be exponentially stable about zero. Consequently, the dynamical network (5) is exponentially synchronized.

Proof: Define a Lyapunov function $V = \mathbf{w}_k^T\mathbf{P}\mathbf{w}_k$. Differentiating V along the error dynamical trajectory (23) and using (22), one obtains

$$\begin{aligned} \dot{V} &= \dot{\mathbf{w}}_k^T\mathbf{P}\mathbf{w}_k + \mathbf{w}_k^T\mathbf{P}\dot{\mathbf{w}}_k \\ &= \left[(\mathbf{A} + \lambda_k\mathbf{LH})\mathbf{w}_k + \mathrm{g}\left(\mathbf{s}(t)\right)\mathbf{w}_k\right]^T\mathbf{P}\mathbf{w}_k \\ &\quad + \mathbf{w}_k^T\mathbf{P}\left[(\mathbf{A} + \lambda_k\mathbf{LH})\mathbf{w}_k + \mathrm{g}\left(\mathbf{s}(t)\right)\mathbf{w}_k\right] \\ &= \mathbf{w}_k^T\left((\mathbf{A} + \lambda_k\mathbf{LH})^T\mathbf{P} + \mathbf{P}(\mathbf{A} + \lambda_k\mathbf{LH})\right)\mathbf{w}_k \cdot \\ &\quad + 2\left[\mathrm{g}\left(\mathbf{s}(t)\right)\mathbf{w}_k\right]^T\mathbf{P}\mathbf{w}_k \\ &\le \mathbf{w}_k^T\left((\mathbf{A} + \lambda_k\mathbf{LH})^T\mathbf{P} + \mathbf{P}(\mathbf{A} + \lambda_k\mathbf{LH})\right)\mathbf{w}_k \\ &\quad + 2\rho\|\mathbf{w}_k\| \cdot \|\mathbf{P}\mathbf{w}_k\| \end{aligned}$$

Since $2\|\mathbf{w}_k\| \cdot \rho\|\mathbf{P}\mathbf{w}_k\| \le \rho^2\|\mathbf{P}\mathbf{w}_k\|^2 + \|\mathbf{w}_k\|^2$, using (24), one further has

$$\begin{aligned} \dot{V} &\le \mathbf{w}_k^T\left((\mathbf{A} + \lambda_k\mathbf{LH})^T\mathbf{P} + \mathbf{P}(\mathbf{A} + \lambda_k\mathbf{LH})\right)\mathbf{w}_k \\ &\quad + \rho^2\|\mathbf{P}\mathbf{w}_k\|^2 + \|\mathbf{w}_k\|^2 \\ &= \mathbf{w}_k^T(\mathbf{A}^T\mathbf{P} + \mathbf{PA} + \lambda_k\mathbf{PLH} + \lambda_k\mathbf{H}^T\mathbf{L}^T\mathbf{P} \\ &\qquad + \rho^2\mathbf{PP} + \mathbf{I})\mathbf{w}_k \\ &< -2\delta\mathbf{w}_k^T\mathbf{P}\mathbf{w}_k \\ &= -2\delta V. \end{aligned}$$

Based on the Lyapunov stability theory, the error dynamical system (23) is uniformly stable about zero. Consequently, the dynamical network (5) is exponentially synchronized. □

Lemma 2 (Schur Complements [33]): For a given symmetric matrix $\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{21} \\ \mathbf{S}_{12} & \mathbf{S}_{22} \end{bmatrix}$, where $\mathbf{S}_{11} = \mathbf{S}_{11}^T$, $\mathbf{S}_{12} = \mathbf{S}_{21}^T$, and $\mathbf{S}_{22} = \mathbf{S}_{22}^T$, the condition $\mathbf{S} < \mathbf{0}$ is equivalent to

$$\begin{aligned} \mathbf{S}_{22} &< \mathbf{0} \\ \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{12}^T &< \mathbf{0}. \end{aligned}$$

Using Lemma 2, condition (24) can be transformed into the LMI form of

$$\begin{bmatrix} \mathbf{PA}+\mathbf{A}^T\mathbf{P}+2(\delta-\rho)\mathbf{P}+\lambda_k\mathbf{XH}+\lambda_k\mathbf{H}^T\mathbf{X}^T & \rho\mathbf{P}+\mathbf{I} \\ \rho\mathbf{P}+\mathbf{I} & -\mathbf{I} \end{bmatrix} < \mathbf{0}, \qquad k=2,\cdots,N \quad (25)$$

where $\mathbf{X}=\mathbf{PL}$.

If the pair $(\mathbf{A},\mathbf{H})$ is observable, and suitable matrices $\mathbf{X}$ and $\mathbf{P}$ are selected such that (25) is satisfied, the error dynamical system (23) with $\mathbf{L}=\mathbf{P}^{-1}\mathbf{X}$ will be uniformly stable about zero. Consequently, the dynamical network (5) is asymptotically synchronized.

Remark 3: The Lyapunov inequality (24) is generally difficult to be solved directly. By transforming (24) into its LMI form (25) using Schur complements as shown above, a feasible set of $\mathbf{X}$ and $\mathbf{P}$ can be found by using the existing LMI numerical toolbox [26], [33].

IV. Illustrative Example

Here, we illustrate the proposed scheme and the criteria of synchronization with a network example, where each node is a chaotic modified Chua's circuit. Although other nonlinear systems may be applied, this chaotic system is chosen as it is complex but also can be easily conformed to our formulation.

The modified Chua's circuit is described by [23], [24]

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - f_1(x_1)) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta x_2 \end{aligned} \qquad (26)$$

where $f_1(x)$ is a smooth sine-type function in the form of

$$f_1(x) = \begin{cases} \frac{b\pi}{2a}(x-2ac_1), & \text{if } x \geq 2ac_1 \\ -b\sin\left(\frac{\pi x}{2a}+d_1\right), & \text{if } -2ac_1 < x < 2ac_1 \\ \frac{b\pi}{2a}(x+2ac_1), & \text{if } x \leq -2ac_1 \end{cases} \qquad (27)$$

with constants $\alpha>0$, $\beta>0$, $a>0$, and $b>0$. The number of scrolls n is determined by c_1 and d_1, namely

$$n = c_1 + 1 \qquad (28)$$

$$d_1 = \begin{cases} \pi, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases} \qquad (29)$$

For example, by setting the parameters $\alpha=9.5$, $\beta=11$, $a=1.6$, $b=0.10$, $c_1=2$, and $d_1=\pi$, we have $n=3$, and the generated three-scroll attractor is depicted in Fig. 1. The corresponding sine function $f_1(\bullet)$ is shown in Fig. 2.

Referring to (3), a dynamical network with N nodes can be constructed as follows:

$$\begin{aligned} \dot{x}_{i1} &= \alpha(x_{i2} - f_1(x_{i1})) + l_1\sum_{j=1}^{N} c_{ij}y_j \\ \dot{x}_{i2} &= x_{i1} - x_{i2} + x_{i3} + l_2\sum_{j=1}^{N} c_{ij}y_j, \qquad i=1,2,\cdots,N \\ \dot{x}_{i3} &= -\beta x_{i2} + l_3\sum_{j=1}^{N} c_{ij}y_j. \end{aligned} \qquad (30)$$

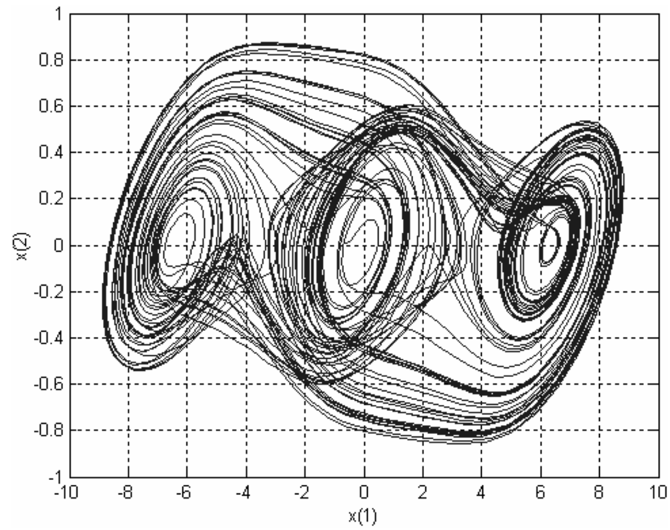

Fig. 1. Chaotic three-scroll attractors on the (x_1, x_2) plane.

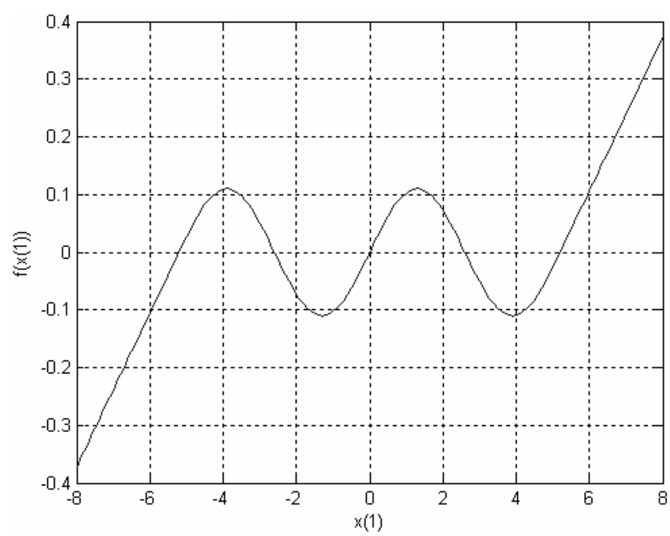

Fig. 2. Sine-type function for generating a chaotic three-scroll attractor.

From system (26), we have

$$\mathrm{f}(\mathbf{x}) = \begin{bmatrix} \alpha(x_2 - f_1(x_1)) \\ x_1 - x_2 + x_3 \\ -\beta x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}$$

$$\mathrm{g}(\mathbf{x}) = \begin{bmatrix} -af_1(x_1) \\ 0 \\ 0 \end{bmatrix}$$

and $\rho=\alpha(b\pi/2a)$ as obtained from Fig. 2 [24].

Two typical coupling configurations, with global coupling and the nearest-neighbor, respectively, are considered below. It should be pointed out that it is much easier to achieve synchronization for the cases where the nodes are (stable) equilibrium points or (orbitally stable) periodic orbits, so they are omitted.

A. Global Coupling

First, we consider the situation of global coupling where any two nodes in the network are connected directly. The corresponding coupling configuration matrix is

$$\mathbf{C} = \begin{bmatrix} -N+1 & 1 & \cdots & 1 \\ 1 & -N+1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -N+1 \end{bmatrix}. \tag{31}$$

Matrix $\mathbf{C}$ has a single eigenvalue at 0, i.e., $\lambda_1 = 0$, while $\lambda_i = -(N-1)$ for $i = 2, 3, \cdots, N$. By choosing $\delta = 0.2$ and $\mathbf{H} = [4\ 5\ 0]$, $(\mathbf{A}, \mathbf{H})$ is observable.

For simplicity, a four-node network $(N = 4)$ is considered. By solving the inequality (25) with the MATLAB LMI toolbox, $\mathbf{L} = [0.3146\ 2.5539\ 14.6589]^T$ can be easily obtained. Fig. 3 shows the behavior of the first state variable of node 1 and chaos synchronization in the four-node network (30) under the global coupling configuration (31).

It can be noticed that, if $N \to \infty$, meaning that the network is large enough, then $\lambda_i \to -\infty$, $i = 2, 3, \cdots, N$. In this case, synchronization in the network (30) can be easily achieved with a very small observer gain matrix $\mathbf{L}$ according to (25), implying that synchronization is very easy to be achieved. This phenomenon was also observed in many diagonally coupling networks [16]–[18].

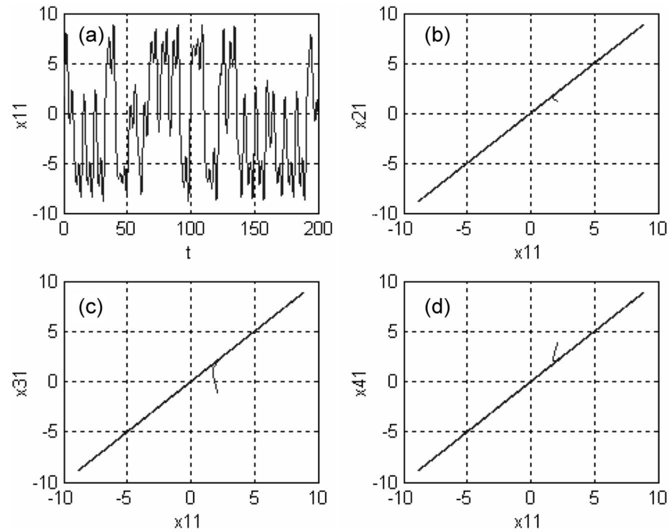

Fig. 3. Synchronization of the four-node network (30) under the global coupling configuration. (a) The behavior of the first state variable of the first node. (b) Synchronization of the first state variables between nodes 1 and 2. (c) Synchronization of the first state variables between nodes 1 and 3. (d) Synchronization of the first state variables between nodes 1 and 4.

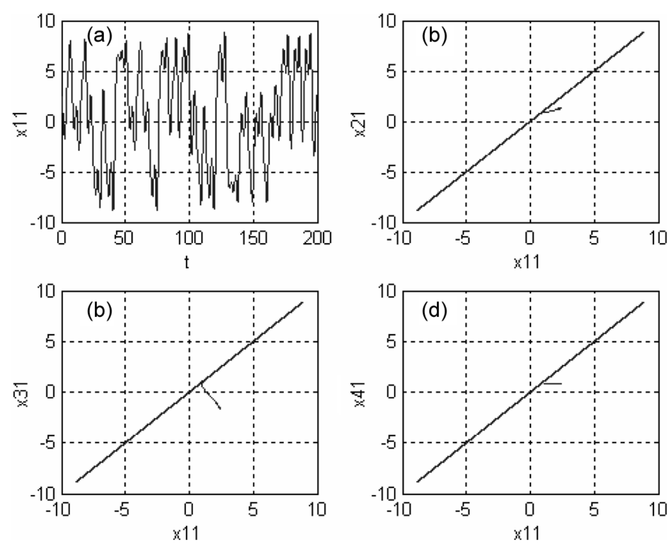

Fig. 4. Synchronization of the four-node network (30) under the nearest-neighbor coupling configuration. (a) The behavior of the first state variable of the first node. (b) Synchronization of the first state variables between nodes 1 and 2. (c) Synchronization of the first state variables between nodes 1 and 3. (d) Synchronization of the first state variables between nodes 1 and 4.

B. Nearest-Neighbor Coupling

The nearest-neighbor coupling configuration consists of nodes arranged in a ring and coupled to the nearest neighbors. In this case, the coupling configuration matrix is

$$\mathbf{C} = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}. \tag{32}$$

The eigenvalues of $\mathbf{C}$ are

$$\left\{ -4\sin^2\left(\frac{k\pi}{N}\right),\ k = 0, 1, \cdots, N-1 \right\}. \tag{33}$$

For simplicity, we also consider a four-node network as an example. Assuming that $\delta = 0.3$ and $\mathbf{H} = [4\ 5\ 0]$, one can obtain $\mathbf{L} = [-6.3757\ 17.7836\ 166.1401]^T$ such that (25) is satisfied, and the behaviors of chaos synchronization in the four-node network (30) under nearest-neighbor coupling are depicted in Fig. 4.

It is noted that, if $N \to \infty$, implying a sufficient large network, the second large eigenvalue $\lambda_2 = -4\sin^2(\pi/N)$ in (33) will tend to zero. In this case, a sufficiently large $\mathbf{L}$ is needed, as shown in (25), to ensure synchronization in network (30). Therefore, synchronization in network (30) is not easily attained if it consists of a large number of nodes. The same phenomenon was also observed in various diagonally coupled networks [16]–[18]. The synchronizability of such a network (with a very large number of nodes) can be improved if additional connections to some far-located nodes are established, leading to a small-world network model [16], [18].

Remark 4: The proposed scheme may be applied to the other types of dynamical networks, such as small-world and scale-free ones. For example, with a small-world network, the eigenvalues λ_k of the coupling configuration matrix $\mathbf{C}$ can be obtained via the numerical simulation [18], and the observer gain $\mathbf{L}$ is obtained by solving the LMI (25).

V. Conclusion

A new scheme has been proposed for synchronization in complex dynamical networks based on the approach of state observer design. The proposed scheme is better than the existing ones for diagonally coupling networks in the sense that it only requires a scalar signal for coupling between any pair of directly connected nodes, which is more practical and convenient to use in real engineering applications. By using the Lyapunov stability theory, some sufficient conditions have been established to ensure synchronization in the networks. The criteria are further transformed to the LMI form, so that suitable observer gains can be easily obtained by using the available LMI toolbox. Two typical dynamical network configurations have been simulated, with global coupling and nearest-neighbor coupling, respectively, where each node is a modified Chua's circuit, which illustrates the effectiveness of the proposed scheme and validates the criteria derived in the paper. This state-observer-based approach can be further generalized and applied to other types of coupling configurations of various complex dynamical networks, which will be carried out in our future research.

References

[1] S. H. Strogatz, "Exploring complex network," *Nature*, vol. 410, pp. 268–276, 2001.

[2] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Rev. Modern Phys.*, vol. 74, pp. 47–97, 2002.

[3] M. Barahona and L. M. Pecora, "Synchronization in small-world systems," *Phys. Rev. Lett.*, vol. 89, no. 5, p. 054 101, 2002.

[4] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, pp. 509–512, 1999.

[5] D. J. Watts and S. H. Strogatz, "Collective dynamics of lsquo;small-world' networks," *Nature*, vol. 393, pp. 440–442, 1998.

[6] L. O. Chua and T. Roska, *Cellular Neural Networks and Visual Computing: Foundation and Applications*. New York: Cambridge Univ. Press, 2002.

[7] K. Kaneko, Ed., *Coupled Map Lattices*. Singapore: World Scientific, 1992.

[8] P. M. Gade, "Synchronization of oscillators with random nonlocal connectivity," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat, Interdiscip. Top.*, vol. 54, no. 1, pp. 64–70, 1996.

[9] P. M. Gade and C. K. Hu, "Synchronous chaos in coupled map lattices with small-world interactions," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat, Interdiscip. Top.*, vol. 62, no. 5, pp. 6409–6413, 2000.

[10] M. Z. Ding and W. M. Yang, "Stability of synchronous chaos and on-off intermittency in coupled map lattices," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat, Interdiscip. Top.*, vol. 56, no. 4, pp. 4009–4016, 1997.

[11] G. Rangarajan and M. Z. Ding, "Stability of synchronized chaos in coupled dynamical systems," *Phys. Lett. A*, vol. 296, pp. 204–209, 2002.

[12] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, no. 8, pp. 821–824, 1990.

[13] C. W. Wu and L. O. Chua, "A unified framework for synchronization and control of dynamical systems," *Int. J. Bifur. Chaos*, vol. 4, pp. 979–998, 1994.

[14] ——, "Synchronization in an array of linearly coupled dynamical systems," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 42, no. 8, pp. 430–447, Aug. 1995.

[15] C. W. Wu, *Synchronization in Coupled Chaotic Circuits and Systems*. Singapore: World Scientific, 2002.

[16] X. Wang and G. Chen, "Complex network: Small-world, scale-free, and beyond," *IEEE Circuits Syst. Mag.*, vol. 3, no. 2, pp. 6–20, Feb. 2003.

[17] ——, "Synchronization in scale-free dynamical networks: Robustness and fragility," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 49, no. 1, pp. 54–62, Jan. 2002.

[18] ——, "Synchronization in small-world dynamical networks," *Int. J. Bifurc. Chaos*, vol. 12, no. 1, pp. 187–192, 2002.

[19] J. Lü, X. Yu, and G. Chen, "Chaos synchronization of general complex dynamical networks," *Physica A*, vol. 334, no. 1–2, pp. 281–302, 2004.

[20] J. Lü, X. Yu, G. Chen, and D. Cheng, "Characterizing the synchronizability of small-world dynamical networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 51, no. 4, pp. 787–796, Apr. 2004.

[21] J. Lü and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 841–846, Jun. 2005.

[22] X. Li and G. Chen, "Synchronization and desynchronization of complex dynamical networks: An engineering viewpoint," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 50, no. 11, pp. 1381–1390, Nov. 2003.

[23] W. K.-S. Tang, G. Q. Zhong, G. Chen, and K. F. Man, "Generation of n-scroll attractors via sine function," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 48, no. 11, pp. 1369–1372, Nov. 2001.

[24] G.-P. Jiang and W. K.-S. Tang, "A global synchronization criterion for coupled chaotic systems via unidirectional linear error feedback approach," *Int. J. Bifurc. Chaos*, vol. 12, no. 10, pp. 2239–2253, 2002.

[25] F. C. M. Lau and C. K. Tse, *Chaos-Based Digital Communication Systems: Operating Principles, Analytical Methods and Performance Evaluation*. Berlin, Germany: Springer-Verlag, 2003.

[26] T.-L. Liao and N.-S. Huang, "An observer-based approach for chaotic synchronization with application to secure communications," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 46, no. 9, pp. 1144–1150, Sep. 1999.

[27] H. Nijmeijer and I. M. Y. Mareels, "An observer looks at synchronization," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 44, no. 10, pp. 882–890, Oct. 1997.

[28] T. Ushio, "Synthesis of synchronized chaotic systems based on observers," *Int. J. Bifurc. Chaos*, vol. 9, no. 3, pp. 541–546, 1999.

[29] J. Alvarez-Ramirez, H. Puebla, and I. Cervantes, "Stability of observer-based chaotic communications for a class of Lur'e systems," *Int. J. Bifurc. Chaos*, vol. 12, no. 7, pp. 1605–1618, 2002.

[30] G. Grassi and S. Mascolo, "Nonlinear observer design to synchronize hyperchaotic systems via a scalar signal," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 44, no. 10, pp. 1011–1014, Oct. 1997.

[31] Ö. Morgül and E. Solak, "On the synchronization of the chaotic systems by using state observers," *Int. J. Bifurc. Chaos*, vol. 7, no. 6, pp. 1307–1322, 1997.

[32] M. J. Ogorzalek, "Taming chaos-part I: Synchronization," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 40, no. 10, pp. 693–699, Oct. 1993.

[33] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[34] P. Erdös and A. Rényi, "On the evolution of random graphs," *Publ. Math. Inst. Hung. Acad. Sci.*, vol. 5, pp. 17–60, 1960.

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